Solitary Waves under the Influence of a Long-Wave Mode

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Abstract

The dynamics of solitons of the nonlinear Schrödinger equation under the influence of dissipative and dispersive perturbations is investigated. In particular a coupling to a long-wave mode is considered using extended Ginzburg-Landau equations. The study is motivated by the experimental observation of localized wave trains ('pulses') in binary-liquid convection. These pulses have been found to drift exceedingly slowly. The perturbation analysis reveals two distinct mechanisms which can lead to a 'trapping' of the pulses by the long-wave concentration mode. They are given by the effect of the concentration mode on the local growth rate and on the frequency of the wave. The latter, dispersive mechanism has not been recognized previously, despite the fact that it dominates over the dissipative contribution within the perturbation theory. A second unexpected result is that the pulse can be accelerated by the concentration mode despite the reduced growth rate ahead of the pulse. The dependence of the pulse velocity on the Rayleigh number is discussed, and the hysteretic 'trapping' transitions suggested by the perturbation theory are confirmed by numerical simulations, which also reveal oscillatory behavior of the pulse velocity in the vicinity of the transition. The derivation and reconstitution of the extended Ginzburg-Landau equations is discussed in detail.

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I. INTRODUCTION

The appearance of stable localized structures in spatially homogeneous extended systems constitutes an exciting feature of various pattern-forming systems. Although the linear instabilities arise from extended modes, various nonlinear mechanisms can lead to the confinement of the structure to a small part of the system, the size of which is independent of the system size.

In systems exhibiting steady structures the competition between structures with two or more different wave numbers can lead to stable domains of one wave number within a background of a different wave number [1–4]. Such structures have been observed in convection experiments in narrow channels [1]. They are expected to be particularly accessible in parametrically driven surface waves in ferrofluids [5]. The stability of domain structures is due to an oscillatory interaction between the domain walls separating the domains [3,4]. Zig-zag structures constitute a two-dimensional analog of these domains [6].

A different class of localized structures has been found in systems exhibiting a parity-breaking bifurcation from a steady structure to traveling waves [7–13]. Instead of extended traveling waves one observes that the parity (reflection symmetry) of the structure is broken only over a stretch of a few wavelengths. Within that part, which is drifting through the systems, the structure is traveling, whereas the remaining structure is motionless. Here the localization is connected with the fact that generically the extended traveling waves are unstable to side-band perturbations right at their onset [11–13]. Remarkably, the parity-breaking bifurcation need not be subcritical for this localization to occur [11,12].

In this paper localized traveling waves are investigated which arise in the convection in binary liquid mixtures. This system has been the subject of extensive experimental and theoretical studies. In quasi-one-dimensional set-ups 'pulses' of traveling waves have been observed with lengths ranging from a few up to 15 wavelengths [14–18]. In sufficiently homogeneous systems they drift extremely slowly through the system [18]. Outside the pulses the fluid is in the conductive, motionless state. In two-dimensional systems such

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pulses have also been observed but were always found to be (very weakly) unstable [19]. A more detailed discussion can be found in a recent review [20] and in [21,22].

Theoretically, the pulses in binary-mixture convection have posed various intriguing problems. These are related to the mechanism of localization, the extremely slow drift of the pulses and their stability. Numerical simulations of the Navier-Stokes equations have provided quite some insight into the physics of the localized (and the extended) waves and in particular into the role of the concentration field [21–24]. In order to get a detailed understanding of the mechanisms leading to the localization of the traveling waves and the slowing-down of the resulting pulse it is highly desirable to study the problem also analytically. However, in particular due to the subcritical nature of the bifurcation to traveling waves, no rigorous asymptotic reduction of the Navier-Stokes equation to a set of simpler evolution equations describing pulses appears available. Nevertheless, asymptotic analysis allows the derivation of equations of the Ginzburg-Landau type which capture and elucidate the essential mechanisms of the system.

Various authors have shown that the dispersive character of the waves can lead to the localization of the waves. In the limit of strong dispersion the complex Ginzburg-Landau equation describing the backward Hopf bifurcation to traveling waves was considered as a dissipatively perturbed nonlinear Schrödinger equation [25–27]. It was shown that the dissipative perturbations could select a stable solitary wave from the continuum of soliton solutions. In the opposite limit of weak dispersion the interaction of fronts connecting the conductive and the convective state were considered. It was shown that dispersion can lead to a spatially oscillatory contribution to the interaction, thus allowing stably bound pairs of fronts forming a solitary wave [28–31].

In both analytical approaches serious qualitative disagreements with the experimental observations remain. First, the theoretical drift velocity of the pulses [32–34] is found to be by a factor of 20 to 40 and more larger than that observed in experiment [18,35]. In particular, the theoretical pulse velocity is always in the same direction as the phase velocity of the waves. Experimentally, however, backward traveling pulses are found as well [35]. It

has been argued that this discrepancy could be eliminated if nonlinear gradient terms were kept in the Ginzburg-Landau equation [36]. This would, however, require that the linear group velocity be compensated for by higher-order correction terms, which would constitute a quite delicate balance. Moreover, within the Ginzburg-Landau equation small and large pulses coexist with the large pulse being stable and the small one always being unstable [28,30]. In recent experiments, however, only a single (stable) pulse was found for strongly negative separation ratio ψ and two pulses for weakly negative ψ. In the latter case it was the shorter pulse which was stable whereas the longer pulse was unstable [35].

As pointed out in previous papers mass diffusion in liquid mixtures is extremely slow as compared to heat diffusion [37]. This introduces an additional slow time scale connected to a concentration mode and leads to the break-down of the complex Ginzburg-Landau equations already for quite small amplitudes of convection. Based on this observation extended Ginzburg-Landau equations were introduced which capture this aspect [37,38]. Numerical simulations of these equations showed that the additional mode can lead to a considerable slow-down of the pulses [37]. An analysis of the interaction of fronts [39] showed that this mode introduces a new mechanism for localization, which is independent of dispersion. Strikingly, it depends strongly on the direction of propagation of the pulse [40]. Numerical simulations of the extended equations revealed that more than two pulse solutions can coexist. In that case the longer pulse can be unstable in agreement with the experimental results [41].

In the present paper I investigate the extended Ginzburg-Landau equations analytically in detail. I consider the strongly dispersive case and study in particular the effect of the concentration mode on the perturbed soliton solution. In certain limits the concentration mode can be eliminated and leads to a change of the coefficients in the Ginzburg-Landau equation. In particular, it introduces nonlinear gradient terms. Most notably, its contribution to the coefficients depends on the velocity of the pulse. This leads to a strongly nonlinear dependence of the pulse velocity on the parameters. The calculation shows that there are two different mechanisms that can lead to a slow-down ('trapping') of the pulse.

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As found in numerical simulations of the Navier-Stokes equations [21] the concentration mode strongly reduces the buoyancy of the liquid ahead of the pulse. This can - but need not - slow down the pulse. There is in fact the possibility that the pulse is sped-up by the concentration mode. This is due to the fact that the variation in buoyancy also leads to a change in the wave number of the pulse which is directly related to its velocity. The second mechanism does not rely on a change in the buoyancy. It arises from the change in frequency of the wave due to the concentration mode and can also induce 'trapping'. This was not anticipated from the numerical simulations [21].

This paper is organized as follows. In sec.II the extended Ginzburg-Landau equations are presented. Their validity and their relevance to other systems exhibiting an interaction between short-wave and long-wave modes is discussed in some detail. The concentration field generated by a pulse within this framework is described in sec.III. The main results for the influence of the concentration mode on the pulse and its velocity are obtained in sec.IV using perturbation theory around the soliton of the nonlinear Schrödinger equation. Numerical simulations supporting the conclusions based on the perturbation result are presented in sec.V. In two appendices a slightly more general derivation of the extended Ginzburg-Landau equation from the Navier-Stokes equations is presented, which leads to quantitative agreement with the usual Ginzburg-Landau equation in a suitable limit. Their self-consistency is confirmed using Ward identities.

II. THE EXTENDED GINZBURG-LANDAU EQUATIONS

As indicated in the introduction, the pulses of the complex Ginzburg-Landau equation differ in various aspects *qualitatively* from those observed experimentally. The numerical simulations of the Navier-Stokes equations indicate that certain modes of the concentration field are independent of the convective amplitude [21]. In the complex Ginzburg-Landau equation the concentration field is, however, adiabatically slaved to the convective amplitude. The concentration field can become an independent dynamic quantity only if its time scales

are slow, i.e. of the same order as those of the convective amplitude. In [37] it was recognized that in binary-liquid convection this is indeed the case since the mass diffusion is by a factor 100 slower than heat diffusion. Since the Ginzburg-Landau equation is only valid as long as the time scales of the convective amplitude are slow as compared to all other time scales, the slow mass diffusion implies that the regime of validity of the Ginzburg-Landau equation is limited to very small amplitudes.

To capture the additional dynamics of the concentration field I have considered the Navier-Stokes equations in an expansion in which the Lewis number, which characterizes the ratio of mass diffusion to heat diffusion, is of the same order as the growth rate of the convective amplitude [37]. In the relevant limit of vanishing mass diffusion two types of critical modes arise: the usual convective mode A, which involves the stream function, the temperature and the concentration field, and an infinite number of critical modes which are all associated with the concentration field alone: any concentration mode which is independent of the horizontal coordinate has vanishing growth rate. Strictly speaking, this implies that in a weakly nonlinear theory this infinite number of modes would have to be kept. However, since all these modes are damped for non-zero Lewis number, it is reasonable to keep only those modes which are driven directly by the convective amplitude.

For free-slip-permeable boundary conditions it turns out that only a single mode C is excited to lowest order. The form of the evolution equation for A and C can then be derived without any detailed calculation and irrespective of the boundary conditions by using the translation and reflection symmetries of the system. One obtains to third order [37]

$$\partial_t A + (s + s_2 C)\partial_x A = d\partial_x^2 A + (a + fC + f_2 C^2 + f_3 \partial_x C)A + cA|A|^2 + ..., \tag{1}$$

$$\partial_t C = \delta\partial_x^2 C - \alpha C + h_2 \partial_x |A|^2 + (h_1 + h_3 C)|A|^2 + ih_4(A^*\partial_x A - A\partial_x A^*) + ... \tag{2}$$

For these boundary conditions the coefficients have been calculated explicitly from the Navier-Stokes equations [38]. Due to the purely advective nature of the nonlinearity in the Navier-Stokes equation, no nonlinear terms of the form C^n arise in (1,2). Since C is even under reflections in x no terms like $\partial_x C^n$ can arise. Note that in [37,38,40] the term

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proportional to h_4 has been given incorrectly. It describes the wave-number dependence of the mixing term $h_1|A|^2$. A somewhat generalized derivation of (1,2) is presented in appendix A.

For realistic boundary conditions more modes may arise. Considering a complete set of vertical eigenfunctions $C^{(n)}(z)$, it is in general to be expected that a whole range of them is driven simultaneously. However, their damping due to vertical diffusion (cf. coefficient α in (2)) grows quadratically with their number n of nodes. In addition, their effect on the convective amplitude depends on the strength of the relevant coupling coefficients (e.g. f in (1)). Since the coupling coefficients are proportional to the projection of the respective concentration mode on the convective amplitude they decrease rapidly with the number of nodes. Thus, high-order modes will have only a weak effect. On that account the most relevant modes will be $C^{(0)}$ and $C^{(1)}$. In addition, the modes excited to leading order will be odd in the vertical coordinate, since the convective mode is even and the nonlinear operator is odd (cf. (A1)). Thus, although the mean concentration mode $C^{(0)}$ experiences no damping at all (it is actually conserved) it may still be less relevant than $C^{(1)}$ since the latter is excited at a lower order.

The above arguments suggest to study extended Ginzburg-Landau equations for a convective amplitude A coupled to a single, weakly damped, odd concentration mode $C^{(1)} \equiv C$. The importance of such an antisymmetric concentration mode has also become apparent in full numerical simulations with realistic boundary conditions [24,21]. Since C is invariant under translations and reflections in the horizontal coordinate it can couple to all terms in the equation for A as demonstrated in (1) to third order. It is expected, however, that it will affect A most strongly through the change of the small linear growth rate, which is one of the expansion parameters in the problem, and the linear frequency *via* the term fCA. A striking effect of the concentration field on the local buoyancy has been identified in [24,21]. In the following I will therefore consider the somewhat simplified equations

$$\partial_t A + s\partial_x A = d\partial_x^2 A + (a + fC)A + c|A|^2 A + p|A|^4 A + g_1 A\partial_x|A|^2 + g_2|A|^2\partial_x A, \qquad (3)$$

$$\partial_t C = \delta \partial_x^2 C - \alpha C + h_1 |A|^2 + h_2 \partial_x |A|^2. \tag{4}$$

This version seems to capture the essential mechanisms of the experimental system. To account for the subcritical nature of the bifurcation fifth-order terms as well as cubic gradient terms have been included[1]. Note that (3,4) have to be considered as reconstituted equations [42]; they are obtained by combining the solvability conditions arising at various orders. Therefore not all terms are of the same order (cf. $h_1|A|^2$ and $h_2\partial_x|A|^2$).

In the analysis I will focus in particular on the effect of the gradient term $h_2\partial_x|A|^2$ which expresses the advection of the concentration field by the traveling wave. As shown by Barten *et al.* the wave generates a concentration current which is antisymmetric with respect to the mid-plane of the convection layer [24,21]. If the convective amplitude varies in space such a current increases the concentration in the top half of the layer, say, and decreases it in the bottom half, thus generating an odd concentration mode C. The other gradient term $ih_4(A^*\partial_x A - A\partial_x A^* c)$ will, however, be neglected; for free-slip-permeable boundary conditions the coupling coefficients h_1 and h_4 are both of the order of the Lewis number and therefore small. Since the term involving h_4 expresses the dependence of the mixing term $h_1|A|^2$ on the wave number it represents a correction to a term which is already small.

Note that the diffusive term $\delta\partial_x^2 C$ is not due to molecular diffusion, which contributes only at higher order (cf. (A25)); instead, it arises through the effect of large-scale variations in the concentration field on the local buoyancy of the fluid which generate vorticity which in turn advects the basic (linear) concentration profile.

It should be emphasized that the nonlinear coefficients of the equation for A are not the same as those in the usual Ginzburg-Landau equation since the latter is obtained from (1,2) in the limit of large Lewis number ($|a| \ll \alpha$) by eliminating C adiabatically. This leads to additional contributions to the coefficients of (1). The elimination is, however, not

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[1]In [37] no quintic terms had been displayed although the saturating term $p|A|^4A$ had been used in the numerical simulations.

completely trivial. If one were to take only the terms to the order given in (2) one would not recover the usual values of the coefficients (not even their limit of vanishing Lewis number, cf. (10)). This is due to the fact that terms of higher order in the Lewis number, which have been dropped in (2), become relevant. In appendix A a derivation of (1,2) is given in which those higher-order terms are retained. The adiabatic elimination of C from these equations leads then to the usual coefficients in the Ginzburg-Landau equation.

The extended Ginzburg-Landau equations (1,2) are not only relevant for binary-mixture convection. They describe quite generally the interaction of an unstable short-wave mode with a stable long-wave mode. Such an interaction arises, for instance, in oscillatory convection with a free surface or for waves on the interface of two immiscible fluids in a channel. There the interface position represents the long-wave mode. The latter case has been investigated in detail in [43]. The interaction of short- and long-wave modes is also important in waves traveling on propagating fronts. There the front position represents the long-wave mode. In the presence of translation symmetry its position itself is irrelevant for the dynamics; only its gradients can enter the equations. In contrast to the case discussed here, the relevant long-wave mode $V \equiv \partial_x C$ transforms then like $V \rightarrow -V$ under reflections in x [44]. In spirit, the extended Ginzburg-Landau equations (1,2) are related to those describing patterns in the presence of Galilean invariance [45] or coupled to a weakly damped mean flow [46–48].

III. THE CONCENTRATION MODE GENERATED BY A PULSE

To lowest order the equation for the concentration mode is a linear diffusion-advection equation with damping and the concentration mode is driven by an inhomogeneous term. It is therefore given by the integral over a suitable Green's function. Unfortunately, this integral appears to have no simple closed-form solution. In this section two limiting cases are discussed.

In order to focus on solutions with a steady envelope the extended Ginzburg-Landau

equations are written in a frame moving with the pulse,

$$\partial_t A + (s - V)\partial_x A = d\partial_x^2 A + (a + fC)A + c|A|^2 A + p|A|^4 A + g_1 A\partial_x|A|^2 + g_2|A|^2\partial_x A \quad (5)$$

$$\partial_t C - V\partial_x C = \delta\partial_x^2 C - \alpha C + h_1|A|^2 + h_2\partial_x|A|^2 \quad (6)$$

For steady solutions of (6) the Green's function $G(x, x')$ is given by

$$G(x, x') = \frac{1}{\sqrt{V^2 + 4\alpha\delta}} \left(e^{-k_1(x-x')}\Theta(x - x') + e^{k_2(x-x')}\Theta(-(x - x')) \right) \quad (7)$$

with

$$k_{1,2} = \frac{1}{2\delta} \left(\sqrt{V^2 + 4\alpha\delta} \pm V \right) \quad (8)$$

and $\Theta(x)$ denoting the Heaviside function. The concentration mode can therefore be written
as

$$C(x) = \int_{-\infty}^{\infty} G(x, x') \left(h_1|A(x')|^2 + h_2\partial_x|A(x')|^2 \right) dx' \quad (9)$$

$$= \frac{1}{\sqrt{V^2 + \alpha\delta}} \int_0^{\infty} (h_1 + h_2 k_2)e^{-k_2 x'}|A(x + x')|^2 + (h_1 - h_2 k_1)k_1 e^{-k_1 x'}|A(x - x')|^2 dx'.$$

For the perturbation calculation in sec.IV this integral will have to be evaluated for $|A(x)| = \lambda\mathrm{sech}(\lambda x) + h.o.t.$ Since I was not able to obtain a closed form for that integral I discuss two
limiting cases.

For large α the decay rates k_i become large and it is sufficient to expand $|A(x - x')|^2$
around x. One then obtains

$$C(x) = \frac{1}{\sqrt{V^2 + 4\alpha\delta}} \sum_{n=0}^{\infty} \partial_x^n|A(x)|^2 \left((h_1 + h_2 k_2)\frac{1}{k_2^{n+1}} - (h_1 - h_2 k_2)\frac{1}{(-k_1)^{n+1}} \right)$$

$$= \frac{h_1}{\alpha}|A|^2 + \left(\frac{h_1 V}{\alpha^2} + \frac{h_2}{\alpha} \right) \partial_x|A|^2$$

$$+ \left(\frac{h_2 V}{\alpha^2} + \frac{h_1}{\alpha}(\frac{V^2}{\alpha^2} + \frac{\delta}{\alpha}) \right) \partial_x^2|A|^2 + \frac{h_2}{\alpha} \left(\frac{V^2}{\alpha^2} + \frac{\delta}{\alpha} \right) \partial_x^3|A|^2 + O(\frac{1}{\alpha^4}). \quad (10)$$

Thus, the concentration mode changes the coefficients of the nonlinear gradient terms in
the equation for the convective amplitude A (or introduces such terms if they should have
been absent). It is noteworthy that the contributions to these terms depend on the velocity

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of the pulse. In this limit, however, the concentration mode can follow adiabatically the convective amplitude and one obtains essentially the usual Ginzburg-Landau equation.

Another, more interesting limit is that of large velocity (or equivalently small α and small δ). In this case the time scale set by the velocity of the pulse is short compared to the relaxation time of the concentration mode. The concentration mode can therefore not follow adiabatically. As a consequence the spatial decay rate corresponding to the decay ahead of the pulse is large whereas that describing the decay of the concentration mode behind the pulse is small. For $V > 0$ the rates are given by

$$k_1 = \frac{V}{\delta}\left(1 + \frac{\alpha\delta}{V^2} + O(V^{-4})\right), \qquad k_2 = \frac{\alpha}{V} + O(V^{-3}). \tag{11}$$

The slow decay behind the pulse allows an expansion of the exponential and one obtains

$$C(x) = \frac{1}{\sqrt{V^2+4\alpha\delta}}\left\{(h_1 - h_2 k_2)\sum_{n=0}^{\infty}\frac{-1}{(-k_1)^n}\partial_x^n|A(x)|^2 + \right.$$
$$\left. (h_1 + h_2 k_2)\sum_{n=0}^{\infty}(-k_2)^n\int_0^{\infty} x'^n|A(x+x')|^2\,dx'\right\}. \tag{12}$$

For $V < 0$ one has

$$k_1 = -\frac{\alpha}{V} + O(V^{-3}), \qquad k_2 = -\frac{V}{\delta}\left(1 + \frac{\alpha\delta}{V^2} + O(V^{-4})\right) \tag{13}$$

and

$$C(x) = \frac{1}{\sqrt{V^2+4\alpha\delta}}\left\{(h_1 - h_2 k_1)\sum_{n=0}^{\infty}(-k_1)^n\int_0^{\infty} x'^n|A(x-x')|^2\,dx' + \right.$$
$$\left. (h_1 + h_2 k_2)\sum_{n=0}^{\infty}\frac{1}{(k_2)^{n+1}}\partial_x^n|A(x)|^2\right\}. \tag{14}$$

After inserting the expansions for k_i into (14) one obtains for the concentration mode

$$C(x) = \left(\frac{h_1\delta}{V^2} - \frac{h_2}{V}\right)|A(x)|^2 + \frac{h_2\delta}{V^2}\partial_x|A(x)|^2 + \left(\frac{h_2\alpha}{V^2} + \frac{h_1}{V}\right)S\int_0^{\infty}|A(x+Sx')|^2\,dx'$$
$$-\frac{h_1\alpha}{V^2}\int_0^{\infty} x'|A(x+Sx')|^2\,dx' + O(\frac{1}{V^3}) \tag{15}$$

with $S = |V|/V$. This expansion is not valid uniformly in space since the gradients in $C(x)$ become small far behind the soliton without $C(x)$ itself being small. The damping term

therefore becomes important and ensures that $C(x)$ goes to 0 sufficiently far behind the pulse. This can be taken care of by a matching procedure. For the perturbation calculation below this is, however, not necessary since the soliton amplitude decays exponentially there. Since the envelope of the soliton of the unperturbed nonlinear Schrödinger equation is even, the odd contributions to C will affect the velocity of the pulse. In particular, for $h_1 = 0$ one expects a relationship between the group velocity s and the pulse velocity V of the form

$$V = s + \frac{h_2}{V^2}K + O(V^{-4}), \tag{16}$$

where the coefficient K depends on the amplitude of the pulse. This is confirmed by the perturbation analysis below in which K is calculated in detail. Thus, in the presence of the concentration mode the pulse velocity is a strongly nonlinear function of the linear group velocity s and of the other coefficients of the extended Ginzburg-Landau equation. Eq.(16) suggests even the possibility of a multi-valued connection between V and s and a hysteretic transition between the different solution branches. This rich behavior is to be contrasted with that described by the complex Ginzburg-Landau equation alone for which the pulse velocity is strictly linear in s even if nonlinear gradient terms are included [36].

IV. THE SOLITON EVOLUTION EQUATIONS

A. Perturbed Nonlinear Schrödinger Equation

In this section the evolution equations for the solitary wave are derived by considering it as a perturbed soliton. The nonlinear Schrödinger equation has four continuous symmetries: translations in space, phase shifts, scaling of the amplitude and Galileian invariance. The soliton solutions therefore form 4-parameter families of solutions characterized by their location x_0, phase ϕ_0, amplitude λ and wave number q. Under small dissipative and dispersive perturbations, as they are introduced by the extended Ginzburg-Landau equations (3,4), the dynamics is predominantly within one of these families and can be characterized

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by the slow evolution of $x_0(t)$, $\phi_0(t)$, $\lambda(t)$ and $q(t)$. The corresponding evolution equations are obtained from solvability conditions arising in the perturbation expansion. In the nonlinear Schrödinger equation these are due not only to proper but also to generalized zero-eigenvectors of the linearized operator in question [49,50].

Thus, I consider the perturbed nonlinear Schrödinger equation [25–27,49–51]

$$i\partial_t A + i(s - V)\partial_x A + \frac{1}{2}\partial_x^2 A + |A|^2 A = \epsilon P, \tag{17}$$

where P contains the remaining terms of the extended Ginzburg-Landau equations including the contribution from the concentration mode through eq.(15). Since the explicit expressions for the concentration mode were only obtained in a frame in which the soliton is at rest, eq.(17) has been transformed to a frame moving with velocity V and it is this velocity that needs to be determined. The amplitude A is expanded around the single-soliton solution,

$$A(x,t) = (A_0(\Theta) + \epsilon A_1(\Theta, T) + ...) \, e^{iq(T)\theta + i\phi}, \tag{18}$$

with

$$A_0 = \lambda(T)\text{sech}(\Theta), \qquad \Theta = \lambda(T)\theta \tag{19}$$

and

$$\partial_t \theta = -v(T), \qquad \partial_x \theta = 1, \qquad \partial_t \phi = \omega(T), \qquad \partial_x \phi = 0. \tag{20}$$

The quantities characterizing the soliton within the family are allowed to vary slowly in time in order to eliminate the secular terms arising in the perturbation expansion. In addition, they - as well as V - are expanded in ϵ,

$$\lambda = \lambda_0 + \epsilon \lambda_1 + ..., \qquad q = q_0 + \epsilon q_1 + ..., \qquad v = v_0 + \epsilon v_1 + ..., \tag{21}$$

$$\omega = \omega_0 + \epsilon \omega_1 + ..., \qquad V = V_0 + \epsilon V_1 + ... \tag{22}$$

Note that the perturbation P can also change the width of the soliton. Therefore a long-wave theory, in which Θ is a slowly varying function of x and t is not sufficient [52]. In the

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present ansatz any time-dependence of λ and q leads to diverging time derivatives of the phase for large values of θ. This poses, however, no problem here since the (bright) soliton goes to 0 exponentially fast there. Inserting the ansatz (18)-(22) into (17) yields at lowest order

$$v_0 = s - V_0 + q_0, \quad \omega_0 = \frac{1}{2}(\lambda_0^2 + q_0^2). \tag{23}$$

Thus, to this order the amplitude and the wave number as well as the velocity and the frequency of the pulse are undetermined. At $O(\epsilon)$ one obtains

$$i\partial_t A_1 + \mathcal{L} A_1 \equiv i\partial_t A_1 - \frac{1}{2}\lambda_0^2 A_1 + \frac{1}{2}\lambda_0^2 \partial_\Theta^2 A_1 + A_0^2(2A_1 + A_1^*) = I, \tag{24}$$

where the inhomogeneity I contains in addition to P the contributions from the derivatives of the slowly varying quantities. The operator $i\mathcal{L}$ is singular. It has two proper zero-eigenvectors iA_0 and $\partial_\Theta A_0$, which arise from the translation and phase shift symmetry of the unperturbed nonlinear Schrödinger equation. In addition, $i\mathcal{L}$ has two generalized eigenvectors $i\Theta A_0$ and $\Theta\partial_\Theta A_0 + A_0$ [49,50],

$$i\mathcal{L}\left\{i\Theta A_0\right\} = -\lambda_0^2 \partial_\Theta A_0, \quad i\mathcal{L}\left\{\Theta\partial_\Theta A_0 + A_0\right\} = i\lambda_0^2 A_0, \tag{25}$$

which arise from the scale and the Galileian invariance. In general, A_1 will contain secular terms. Weinstein has shown [49] that A_1 remains bounded for times of $O(1/\epsilon)$ if I is orthogonal to the generalized null-space of $i\mathcal{L}$, in which case A_1 is also orthogonal to that space. To project iI onto that null-space an appropriate scalar product is defined by

$$(A, B) = Re\left(\int_{-\infty}^{\infty} A^*(\Theta) B(\Theta) d\Theta\right). \tag{26}$$

It corresponds to the usual scalar product on $\mathcal{R}^2$ once the complex quantities are separated in real and imaginary parts. Using the appropriate left eigenvectors, the four solvability conditions are given by

$$Re \int_{-\infty}^{\infty} iA_0 I \, d\Theta = 0, \tag{27}$$

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$$Re \int_{-\infty}^{\infty} \partial_\Theta A_0 I \, d\Theta = 0, \tag{28}$$

$$Re \int_{-\infty}^{\infty} i\Theta A_0 I \, d\Theta = 0, \tag{29}$$

$$Re \int_{-\infty}^{\infty} (A_0 + \Theta \partial_\Theta A_0) I \, d\Theta = 0. \tag{30}$$

Inserting the expansion (18) one obtains

$$\frac{d}{dT}\lambda_0 = \int_{-\infty}^{\infty} \text{sech}(\Theta) P_i \, d\Theta \tag{31}$$

$$\frac{d}{dT}q_0 = -\int_{-\infty}^{\infty} \text{sech}(\Theta) \tanh(\Theta) P_r \, d\Theta \tag{32}$$

$$v_1 = q_1 - V_1 + \frac{1}{\lambda_0^2} \int_{-\infty}^{\infty} \Theta \, \text{sech}(\Theta) P_i \, d\Theta \tag{33}$$

$$\omega_1 = \lambda_0 \lambda_1 + q_0(V_1 + v_1)$$

$$-\frac{1}{\lambda_0} \int_{-\infty}^{\infty} \text{sech}(\Theta) P_r \, d\Theta + \frac{1}{\lambda_0} \int_{-\infty}^{\infty} \Theta \, \text{sech}(\Theta) \tanh(\Theta) P_r \, d\Theta \tag{34}$$

with $P = P_r + iP_i$. Eqs.(31,32) yield evolution equations for the lowest-order contributions to the amplitude and to the wave number. This determines also the velocity and frequency to lowest order (cf. (23)). The quantity of most interest in the present context is the effect of the concentration mode on the velocity of the soliton. To lowest order it is determined by the group velocity s and the wave number q_0, $V_0 = s + q_0$. The relevant question is therefore how the perturbations affect the wave number. It turns out that spatial variations in the growth rate due to the concentration mode, which were also identified in the numerical simulations of the Navier-Stokes equations, lead to $q_0 = 0$ independent of their strength. Therefore, no change in velocity arises at this order. One then has to determine V_1 which in turn depends on q_1 (cf. (33)). This contribution to the wave number is, however, not determined until $O(\epsilon^2)$. Spatial variations in the frequency on the other hand affect the wave number already at lower order.

In order to go to $O(\epsilon^2)$ one needs to determine A_1. This can be done by variation of constants using the homogeneous solutions of the real and the imaginary part of (24), respectively,

$$A_{1r}^{(h)} = \text{sech}(\Theta) \tanh(\Theta), \tag{35}$$

$$A_{2r}^{(h)} = \Theta \operatorname{sech}(\Theta) \tanh(\Theta) + \frac{1}{3} \cosh(\Theta) - \operatorname{sech}(\Theta), \tag{36}$$

$$A_{1i}^{(h)} = \operatorname{sech}(\Theta), \tag{37}$$

$$A_{2i}^{(h)} = \Theta \operatorname{sech}(\Theta) + \sinh(\Theta). \tag{38}$$

The general solution is then given by

$$A_1 = (F_{1r} + G_{1r})A_{1r}^{(h)} + (F_{2r} + G_{2r})A_{2r}^{(h)} +$$
$$i\left((F_{1i} + G_{1i})A_{1i}^{(h)} + (F_{2i} + G_{2i})A_{2i}^{(h)}\right) \tag{39}$$

with

$$F_{1r} = \frac{3}{\lambda_0^2} \int I_r A_{2r}^{(h)} \, d\Theta, \tag{40}$$

$$F_{2r} = \frac{-3}{\lambda_0^2} \int I_r A_{1r}^{(h)} \, d\Theta, \tag{41}$$

$$F_{1i} = \frac{1}{\lambda_0^2} \int I_i A_{2i}^{(h)} \, d\Theta, \tag{42}$$

$$F_{2i} = \frac{-1}{\lambda_0^2} \int I_i A_{1i}^{(h)} \, d\Theta, \tag{43}$$

and $I = I_r + iI_i$. The coefficients G_2 allow the elimination of certain exponential divergences of A_1 for large $|\Theta|$. The solvability conditions arising from the projection onto the proper zero-eigenvectors (31,32) eliminate the other exponential divergences. The contributions from the coefficients G_1 lead only to a fixed shift of the position and of the phase of the soliton. They can therefore be set to 0. After inserting A_1 into (17) the required equations for λ_1 and q_1 are obtained by applying the solvability conditions at $O(\epsilon^2)$.

B. First-Order Evolution Equations: Effect of C on the Frequency

The extended Ginzburg-Landau equations (3,4) lead to the perturbation terms

$$P = i(d_r \partial_x^2 A + a_r A + c|A|^2 A + p|A|^4 A + g_1 A \partial_x |A|^2 + g_2 |A|^2 \partial_x A + f A C). \tag{44}$$

Here C is given for α and δ small as compared to V by

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$$C = \left(-\frac{h_2}{V} + \frac{h_1\delta}{V^2}\right)|A|^2 + \frac{h_2\delta\lambda}{V^2}\partial_\Theta|A|^2 +$$

$$\left(\frac{\alpha h_2}{V^2\lambda} + \frac{h_1}{V\lambda}\right)\int_\Theta^{S\Lambda\infty}|A(\Theta')|^2 d\Theta' + \frac{h_1\alpha}{V^2\lambda^2}\int_\Theta^{S\Lambda\infty}(\Theta - \Theta')\,|A(\Theta')|^2\,d\Theta' \qquad (45)$$

with $\Lambda = |\lambda|/\lambda$. In view of other possible applications of this perturbation expansion a slightly more general perturbation is used

$$P = i(d_r\partial_x^2 A + a_r A + c|A|^2 A + p|A|^4 A + (\rho_1 + i\sigma_1)A\partial_x|A|^2 + (\rho_2 + i\sigma_2)|A|^2\partial_x A +$$

$$(\rho_0 + i\sigma_0)A\frac{1}{\lambda}\int_\Theta^{S\Lambda\infty}|A(\Theta')|^2 d\Theta' +$$

$$(\rho_{-1} + i\sigma_{-1})A\frac{1}{\lambda^2}\int_\Theta^{S\Lambda\infty}(\Theta - \Theta')\,|A(\Theta')|^2\,d\Theta'. \qquad (46)$$

Insertion of this perturbation in eqs.(31)-(34) leads to

$$\frac{d}{dT}\lambda_0 = \frac{16\,\lambda_0^5 p_r}{15} + \left(\frac{4\,c_r}{3} - \frac{2\,d_r}{3} - \frac{4\,\sigma_2 q_0}{3}\right)\lambda_0^3 + 2S\Lambda\,\lambda_0^2\rho_0 + \left(2\,a_r - 2\,d_r\,q_0^2 - 2\,\rho_{-1}\right)\lambda_0, \quad (47)$$

$$\frac{d}{dT}q_0 = S\Lambda\lambda_0\sigma_{-1} - \frac{4\,d_r\,\lambda_0^2 q_0}{3} - \frac{2\,\lambda_0^2\sigma_0}{3} - \left(\frac{4\,\sigma_2}{15} + \frac{8\,\sigma_1}{15}\right)\lambda_0^4, \qquad (48)$$

$$v_1 = q_1 - V_1 - \left(\frac{\rho_2}{3} + \frac{2\,\rho_1}{3}\right)\lambda_0^2 - \rho_0 + \Lambda\frac{\rho_{-1}\pi^2}{6\,\lambda_0}, \qquad (49)$$

$$\omega_1 = \frac{8\,\lambda_0^4 p_i}{9} + (c_i + \rho_2 q_0)\,\lambda_0^2 + (\lambda_1 + \Lambda S\sigma_0)\,\lambda_0 + q_0(v_1 + V_1) - \frac{\sigma_{-1}}{2}. \qquad (50)$$

Eqs.(47,48) constitute two coupled nonlinear evolution equations for the amplitude and the wave number of the pulse. This could in general lead to interesting dynamics. The quadratic term $sS\Lambda\lambda_0^2\rho_0$ in (47) indicates that the subcritical pitchfork bifurcation is perturbed to a transcritical bifurcation.

Here I focus on a discussion of the influence of the concentration mode. On the one hand this restricts the choice of the coefficients $\sigma_j = f_i\rho_j$, $j = -1, ...2$. On the other hand, the coefficients depend on the velocity V of the pulse (cf. (15)) which in turn depends on the wave number. This introduces additional complexity. Explicitly, one obtains

$$\frac{d}{dT}\lambda_0 = \frac{16\,p_r\lambda_0^5}{15} + \frac{4}{3}\left(\left(-\frac{h_2}{V_0} + \frac{h_1\delta}{V_0^2}\right)f_r + c_r - \frac{1}{2}d_r - g_{2,i}q_0\right)\lambda_0^3$$

$$+2\,\Lambda\,S\left(\frac{\alpha\,h_2}{V_0^2} + \frac{h_1}{V_0}\right)f_r\lambda_0^2 + 2\left(a_r - d_r q_0^2 - \frac{f_r h_1\alpha}{V_0^2}\right)\lambda_0 + O(V_0^{-3}), \qquad (51)$$

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$$\frac{d}{dT}q_0 = -\frac{4}{15}\left(\frac{2\,h_2\delta}{V_0^2}f_i + g_{2,i} + 2\,g_{1,i}\right)\lambda_0^4 -$$
$$\frac{2}{3}\left(\left(\frac{h_1}{V_0} + \frac{\alpha\,h_2}{V_0^2}\right)f_i + 2\,d_r q_0\right)\lambda_0^2 + \frac{\lambda_0 f_i h_1 \alpha\,\Lambda\,S}{V_0^2} + O(V_0^{-3}), \tag{52}$$

$$v_1 = q_1 - V_1 - \left(\frac{1}{V_0} - \frac{\Lambda\,S\alpha\,\pi^2}{6\,\lambda_0 V_0^2}\right)f_r h_1 - \frac{(2\,\lambda_0^2\delta + 3\,\alpha)\,f_r h_2}{3\,V_0^2} -$$
$$\frac{1}{3}\left(2\,g_{1,r} + g_{2,r}\right)\lambda_0^2 + O(V_0^{-3}), \tag{53}$$

$$\omega_1 = \left(\frac{\lambda_0\Lambda\,S}{V_0} + \frac{2\,\lambda_0^2\delta - \alpha}{2\,V_0^2}\right)f_i h_1 + \left(-\frac{\lambda_0^2}{V_0} + \frac{\lambda_0\Lambda\,S\alpha}{V_0^2}\right)f_i h_2 +$$
$$q_0(v_1 + V_1) + g_{2,r}\lambda_0^2 q_0 + \lambda_0\lambda_1 + \frac{8\,\lambda_0^4 p_i}{9} + O(V_0^{-3}). \tag{54}$$

Eqs.(51,52) show that λ_0 and q_0 and therefore also V_0 are $O(1)$-quantities and can vary over an $O(1)$-range. The smallness of the perturbations expresses itself only in the slowness of the dynamics. Note, however, that the solvability conditions are only valid in a frame of reference in which the pulse is steady, i.e. $v = 0$, $\frac{d}{dT}\lambda = 0$ and $\frac{d}{dT}q = 0$. This limitation arises from the expansion for the concentration mode (15).

Since $q_0 = V_0 - s$ eq.(52) can be viewed as an equation for V_0. Solving for the group velocity s one obtains

$$s = V_0 + \frac{f_i h_1}{2\,d_r}\left(\frac{1}{V_0} - \frac{3}{2}\frac{S\Lambda\alpha}{\lambda_0 V_0^2}\right) + f_i h_2\frac{1}{V_0^2 d_r}\left(\frac{1}{2}\alpha + \frac{2}{5}\lambda_0^2\delta\right) +$$
$$\frac{1}{5 d_r}(2g_{1i} + g_{2i})\lambda_0^2 + O(V_0^{-3}). \tag{55}$$

Thus, the perturbation terms arising from the dynamical equation for C lead to a strongly nonlinear connection between s and the pulse velocity V_0 ([2]). This is to be contrasted with the result one would obtain from a complex Ginzburg-Landau equation alone in which the coefficients of the nonlinear gradient terms are fixed [36],

$$s = V_0 - \frac{3\,S\Lambda\sigma_{-1}}{4\,\lambda_0 d_r} + \frac{\sigma_0}{2\,d_r} + \frac{1}{5}\frac{\lambda_0^2}{d_r}(\sigma_2 + 2\,\sigma_1). \tag{56}$$

In (56) the perturbation terms lead only to a fixed shift in the velocity (as do the terms

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[2]Note that to leading order in $1/V_0$ the amplitude λ_0 is independent of s and V_0.

involving g_j in (55)). Although the shift depends on the amplitude λ_0 it does not give a natural explanation why V_0 should be close to 0 over a range of parameters.

By contrast, both the terms proportional to h_1 and to h_2 in (55) are suggestive of a range of parameters with small V_0. This is indicated in fig.1 where the contributions to $V_0(s)$ from the h_1-term (dashed line) and from the h_2-term (solid line) are sketched separately for $f_i h_1 > 0$ and $f_i h_2 > 0$ [38]. Of course, the singularities at $V_0 = 0$ are unphysical and arise from a break-down of the expansion (15) for small V_0. Strictly speaking, even the saddle-node bifurcations suggested by fig.1 are beyond the present expansion. In that regime the diffusion and the damping term in (6) come into play. It is expected that they will lead to a smooth connection of the two branches of $V_0(s)$ and thus to a whole range of parameters (with $s > 0$) in which V_0 is small. This is confirmed by the numerical simulations discussed in Sec.V below. An analytical description that removes the singularity would require a better approximation of the concentration mode than that given by (15). Obviously, for negative values of $f_i h_1$ and $f_i h_2$ the concentration mode leads to a speed-up of the pulse (for $s > 0$).

For free-slip-permeable boundary conditions it is found that $f_i h_2 > 0$ and $f_i h_1 > 0$ (cf. (A22,A26)). Thus, within that approximation the extended Ginzburg-Landau equations yield a slow-down of the pulse, as is observed in experiments. For realistic boundary conditions the coefficients in eqs.(3,4) are likely to be different[3]. However, to leading order in the expansion in $1/V_0$ only the sign of the two products $f_i h_1$ and $f_i h_2$ is relevant to determine whether the pulse is slowed down. The other coefficients determine only the strength of the effect and whether the pulse velocity shows hysteresis. Thus, the mechanism is quite robust and is likely to be relevant in the experimental system. It arises from the effect of the concentration mode on the local frequency of the wave through the term $f_i CA$. The spatial variation of the concentration mode implies a spatial variation of the frequency which leads

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[3]Note that the nonlinear coefficients in (3) differ from those of the complex Ginzburg-Landau equation derived in [32] (cf. appendix A).

to a differential phase winding. Since the gradient term $h_2\partial_x|A|^2$ as well as the advective term $V\partial_x C$ induce an odd contribution to C, the phase winding results in a change of the average wave number q_0 and as a consequence in a change in velocity.

Through the wave number the perturbation induces also a frequency shift $\omega_0 = \lambda_0^2/2 + q_0^2/2 - q_0 v_0$ (as compared to the Hopf frequency) of the pulse in the lab frame. It is worth noting that this change in frequency differs from the frequency shift $\omega_{TW} = \lambda_0^2 - q_0^2/2 - sq_0$ of an *extended* wave at the same wave number. Whether this difference is relevant for the fact that the pulse frequencies in the experiments [53] and in the numerical simulations [21] are larger than those of the extended traveling waves is not clear at this point.

C. Second-Order Evolution Equations: Effect of C on the Growth Rate

In earlier work Barten *et al.* had pointed out that the experimentally observed slowing-down of the pulse could be due to the fact that the concentration field leads to a reduction of the local *growth rate* of the convective amplitude ahead of the pulse [21–24]. This observation motivated the derivation of the extended Ginzburg-Landau equations (3,4) [37]. Eq.(55) shows, however, that changes in the local growth rate do not affect the velocity at leading order. Variations in the growth rate come in only at $O(\epsilon)$ (cf. (53)). At that order any change in the wave number q_1 becomes relevant as well. Since q_1 is determined only at $O(\epsilon^2)$ this implies that one has to go to next order in the perturbation expansion in order to capture this effect.

To simplify the calculation it is now assumed that the leading-order, dispersive effect on the wave number vanishes, i.e. $\sigma_j = 0$, $j = -1, ..2$. In addition, since h_1 is small for free-slip-permeable boundary conditions it is neglected as well (thus, $\rho_{-1} = 0$). Once the solvability conditions at $O(\epsilon)$ are met, the amplitude A_1 is determined using the general solution (39). Inserting it into the solvability conditions one then obtains at $O(\epsilon^2)$

$$\frac{d}{dT}\lambda_1 = \left(\frac{16}{3}p_r\lambda_0^4 + 2(2c_r - d_r)\lambda_0^2 + 4\Lambda S\rho_0\lambda_0 + 2(a_r - d_r q_0^2)\right)\lambda_1$$

$$-4d_r q_0 \lambda_0 q_1 + \frac{2176}{945}\lambda_0^7 p_r p_i +$$

$$\frac{32}{15}\lambda_0^5\left((\rho_2 q + 0 + c_i)p_r + \frac{2}{3}(c_r - d_r)p_i\right) + \frac{4}{3}\lambda_0^3\left((c_r - d_r)(q_0\rho_2 + c_i)\right), \tag{57}$$

$$\frac{d}{dT}q_1 = -\frac{4}{3}\lambda_0^2 q_1 d_r - \frac{8}{3}d_r\lambda_1\lambda_0 q_0 + \frac{32}{63}p_r(\rho_1 + \frac{1}{5}\rho_2)\lambda_0^6$$

$$+ \left(\frac{4}{15}(\frac{8}{3}c_r - d_r)\rho_1 + \frac{4}{15}(\frac{1}{3}c_r - d_r)\rho_2 - \frac{128}{45}q_0 p_i d_r + \frac{136}{225}\rho_0 p_r\right)\lambda_0^4$$

$$\frac{4}{3}\lambda_0^3\rho_0\rho_1\Lambda S + \frac{2}{3}\left(\frac{1}{3}(4c_r + d_r)\rho_0 + 2(a_r - d_r q_0^2) - 4(d_r c_i q_0 + \rho_2 d_r q_0^2)\right)\lambda_0^2$$

$$+2\lambda_0\Lambda S\rho_0^2 + 2(a_r - d_r q_0^2)\rho_0. \tag{58}$$

Similar to eqs.(53,54) the equations for v_2 and ω_2 contain λ_2 and q_2 which are only determined at $O(\epsilon^3)$.

For the equations describing the effect of the concentration mode the coefficients ρ_j have to be inserted. Eq.(52) shows that without dispersion $q_0 = 0$ and as a consequence $V_0 = s$. For $s = O(1)$ the equation determining v_1 will be linear in v_1 (cf. (49)) and will therefore not be able to capture the hysteretic behavior found in previous numerical simulations [54]. I therefore take $s = \epsilon s_1$. In order for the concentration field to remain a small perturbation one then has to take $h_2 = O(\epsilon)$ as well. In addition, for the expansion (15) to be valid α and δ have to be small as compared to $V = \epsilon V_1$. With these changes the solvability conditions at $O(\epsilon)$ yield again (51)-(54) with V_1 replaced by $V_1 - s_1$ and V_0 replaced by V_1. At $O(\epsilon^2)$ one obtains

$$\frac{d}{dT}\lambda_1 = \left(4f_r h_2\left(-\frac{\lambda_0^2}{V_1} + \frac{\Lambda S\alpha\lambda_0}{V_1^2}\right) + \frac{16\,p_r\lambda_0^4}{3} + 2(2c_r - d_r)\lambda_0^2 + 2(a_r - d_r q_0^2)\right)\lambda_1$$

$$-4\,d_r q_0 q_1\lambda_0 + \frac{2176\,\lambda_0^7 p_r p_i}{945} + \frac{64}{15}(c_r - d_r)p_i\lambda_0^5$$

$$-\frac{64\,f_r p_i h_2\lambda_0^5}{45\,V_1} + \frac{4\,f_r V_2 h_2\lambda_0^3}{3\,V_1^2} + O(V_1^{-3}), \tag{59}$$

$$\frac{d}{dT}q_1 = -\frac{4}{3}d_r\lambda_0^2 q_1 - \frac{8}{3}d_r\lambda_1\lambda_0 q_0 - \frac{128}{45}q_0 p_i d_r\lambda_0^4$$

$$+ \left(\frac{2}{9}(4c_r + d_r)\lambda_0^2 + 2(a_r - d_r q_0^2) + \frac{136\,p_r\lambda_0^4}{225}\right)\frac{h_2 f_r\alpha}{V_1^2}$$

$$+ \left(\frac{4}{3}(a_r - d_r q_0^2)\lambda_0^2 + \frac{32\,p_r\lambda_0^6}{63} + \frac{4}{15}(\frac{8}{3}c_r - d_r)\lambda_0^4\right)\frac{h_2 f_r\delta}{V_1^2} + O(V_1^{-3}). \tag{60}$$

In the steady case, one obtains from (51,52) $q_0 = 0$ and

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$$a_r = -\frac{8\,p_r\lambda_0^4}{15} + \frac{1}{3}\left(d_r - 2\,c_r + \frac{2\,f_r h_2}{V_1}\right)\lambda_0^2 - \frac{f_r \Lambda\,S\alpha\,h_2\lambda_0}{V_1^2} + O(V_1^{-3}). \qquad (61)$$

This yields for λ_1 and q_1

$$\lambda_1 = -\frac{16}{63}\frac{p_i\lambda_0^3\,(34\,p_r\lambda_0^2 - 21\,d_r + 21\,c_r)}{16\,p_r\lambda_0^2 - 5\,d_r + 10\,c_r} -$$
$$\frac{16}{63}\frac{f_r p_i h_2\lambda_0^3\,(4\,p_r\lambda_0^2 - 105\,d_r)}{(16\,p_r\lambda_0^2 - 5\,d_r + 10\,c_r)^2}\frac{1}{V_1} + O(V_1^{-2}), \qquad (62)$$

$$q_1 = -\frac{f_r h_2}{d_r V_1^2}\left(\frac{26}{75}\,p_r\lambda_0^2 + \frac{1}{3}(c_r - 2d_r)\right)\alpha -$$
$$\frac{f_r h_2}{105 d_r V_1^2}2\,\lambda_0^2\left(8\,p_r\lambda_0^2 + 7(c_r - d_r)\right)\delta + O(V_1^{-3}). \qquad (63)$$

To determine the velocity V_1 the expression for q_1 is inserted into (53) with $h_1 = 0 = v_1$.
Using (61) to replace λ_0^4 one obtains

$$s_1 = V_1 + \left(\left(\frac{26\,p_r\lambda_0^2}{75\,d_r} + \frac{1}{3}\left(1 + \frac{c_r}{d_r}\right)\right)\alpha + \left(\frac{2}{35}\left(11 - \frac{c_r}{d_r}\right)\lambda_0^2 - \frac{2\,a_r}{7\,d_r}\right)\delta\right)\frac{h_2 f_r}{V_1^2}$$
$$+ O(V_1^{-3}). \qquad (64)$$

Since the amplitude λ_0 appears only in the terms of order $O(V_1^{-2})$ it can be replaced by the solution of (51) in the limit $V_1 \to \infty$. In that limit (51) describes an unperturbed backward pitch-fork bifurcation. Of particular interest is the amplitude at the saddle-node bifurcation and at the Hopf bifurcation $(a_r = 0)$,

$$\lambda_{0,SN}^2 = \frac{5(d_r - 2\,c_r)}{16\,p_r} + \frac{5\,f_r h_2}{8\,p_r V_1} + O(V_1^{-3}), \qquad \lambda_{0,H}^2 = 2\lambda_{0,SN}^2. \qquad (65)$$

At these special points the velocity is related to the group velocity s_1 by

$$s_1 = V_{1,SN} + \left(\frac{(4\,c_r + 61\,d_r)}{336\,(-p_r)}(2\,c_r - d_r)\delta + \frac{(14\,c_r + 53\,d_r)}{120}\alpha\right)\frac{h_2 f_r}{d_r V_{1,SN}^2} + O(V_{1,SN}^{-3}), \qquad (66)$$

$$s_1 = V_{1,H} - \left(\frac{(2\,c_r - d_r)}{28\,(-p_r)}(c_r - 11\,d_r)\delta + \frac{(2\,c_r - 11\,d_r)\alpha}{20}\right)\frac{h_2 f_r}{d_r V_{1,H}^2} + O(V_{1,H}^{-3}). \qquad (67)$$

To interpret (64,66,67) note that $p_r < 0$ and $2c_r - d_r > 0$ for the pulse to exist stably in the absence of the concentration mode, $f = 0$ (cf. (61)). Therefore, if $f_r h_2 > 0$ the concentration mode slows the pulse down when it is first created in the saddle-node bifurcation. This

condition, which is satisfied by the coefficients in the free-slip-permeable case [38], corresponds to the concentration field being advected in such a way that the contribution to the buoyancy from the concentration is reduced ahead of the pulse. Eq.(66) supports therefore the physical interpretation for the slow-down given previously [21–24,37].

With increasing a_r the situation can, however, change. Eq.(64) shows that the contribution to the slow-down from the damping term proportional to α decreases with increasing amplitude λ_0^2. The contribution from the diffusive term proportional to δ can have either sign; while the term proportional to a_r always leads to an increase in the velocity with increasing a_r, the effect of the term proportional to λ_0^2 depends on the ratio d_r/c_r. For

$$c_r > 11 d_r \tag{68}$$

its contribution increases the velocity, whereas in the opposite case the 'trapping' is enhanced with increasing amplitude. In the immediate vicinity of the saddle-node bifurcation the amplitude λ_0^2 increases more rapidly than a_r and will dominate the behavior. Thus the outcome depends on the relative strength of damping and diffusion; the pulse velocity will increase if

$$\frac{\alpha}{\delta} > \frac{15}{91} \frac{d_r}{(-p_r)} \left(11 - \frac{c_r}{d_r} \right) \tag{69}$$

and decrease otherwise. Thus, if $c_r > 11 d_r$ the pulse velocity increases with a_r independent of the strength of the damping term and, in fact, for sufficiently large a_r the concentration mode even leads to an over-all *acceleration* of the pulse (cf. (67) at $a_r = 0$), i.e. the pulse is even faster than the linear group velocity. If (69) is satisfied but not (68), i.e. $c_r < 11 d_r$, then the velocity still increases with a_r but remains below the group velocity as long as $a_r < 0$.

These results show that the response of the pulse to the concentration mode is more complex than suggested by the original picture in which the concentration mode only led to a 'barrier' of reduced buoyancy which can 'trap' the pulse [23,37,22]. Since the velocity of the solitary pulse is closely connected to its wave number q (cf. (23)) any effect of

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the concentration mode on the local wave number becomes just as important. In fact, the dependence of the local frequency on the concentration mode, which changes the local wave number, selects a velocity of $O(1)$ (cf. (55)). But even when this dispersive effect is absent ($f_i = 0$) the variation of the concentration mode can select a non-zero wave number [55] somewhat similar to the selection of the wave number by ramps in the control parameter [56,57]. Eq.(53) shows that in the absence of any change in wave number the barrier generated by the advected concentration mode, which is represented by the term proportional to h_2, would always decrease the pulse velocity.

V. NUMERICAL SIMULATIONS

The perturbation analysis in sec.IV B and sec.IV C revealed two distinct mechanisms that can lead to a slow-down of the pulse due to the coupling to the concentration mode. A dispersive mechanism which affects the velocity through the wave number and a mechanism that is based on a suppression of convection ahead of the pulse. The analysis suggests that there is a range of parameters over which the pulse is slow. However, due to the expansion employed for the calculation of the concentration mode the analysis is strictly valid only for large velocities and can only be suggestive with regard to the behavior at small velocities. To go beyond this limitation the extended Ginzburg-Landau equations (3,4) are solved numerically.

To study the dispersive 'trapping', eqs.(3,4) are solved with f purely imaginary. The values of the coefficients are chosen as $d_r = 0.15 + i$, $a = -0.24$, $f = 0.5i$, $c = 2.4 + 2i$, $p = -1.65 + 2i$, $\delta = 0.1$, $\alpha = 0.08$ and the boundary conditions are periodic. Fig.2 shows the velocity of the pulse for $h_1 = 0$, $h_2 = 0.3$ and for $h_1 = 0.5$, $h_2 = 0$. As expected from (55) the pulse is slowed down by the concentration mode (the dotted line gives the velocity without concentration mode, $V = s$). The simulation bears out even the hysteretic transition to a 'trapped' pulse which is suggested by (55) but outside its range of validity. This transition can be understood intuitively by noting that the 'amount of C' generated by the pulse at

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any given location depends on its velocity. If the pulse is slow $C(x)$ becomes large and slows it down even more. If the pulse is fast, however, $C(x)$ is small and the effect on the velocity is small.

The terms $h_1|A|^2$ and $h_2\partial_x|A|^2$ differ in their symmetry and their physical interpretation. Whereas the h_2-term corresponds to the advection of the concentration mode by the wave, the h_1-term represents the generation of C. For the concentration mode to affect the (over-all) wave number it must have an asymmetric contribution. For $v = 0$ the h_1-term leads to a symmetric C-profile. Consequently the wave number is increased on one side of the pulse, but decreased by the same amount on the other side. The over-all wave number and the pulse velocity remain unchanged. For non-vanishing velocity, however, the concentration field lags behind and changes the wave number as illustrated in fig.3. Due to its symmetry this term can induce 'trapping' of the pulse independent of its direction of propagation; for $h_2 = 0$ (55) is invariant under $s \to -s$ and $V_0 \to -V_0$.

By contrast, the h_2-term breaks the reflection symmetry at any velocity. For $f_i h_2 > 0$, the concentration field pushes the pulse to the left independent of its direction of propagation. Thus, for positive group velocity s the pulses are slowed down, whereas for negative s they are accelerated.

If the dispersive effects are small the pulse velocity is affected at next order through the change in the local growth rate given by $f_r CA$. This is the mechanism which was pointed out by Barten *et al.* [21–24]. Some numerical simulations of the extended Ginzburg-Landau equations (3,4) for this case have been presented previously [37,54]. As suggested by (64) the qualitative dependence of the pulse velocity on the group velocity is similar to that due to dispersion. The results of simulations for $a = -0.24$, $d = 0.15+i$, $c = 2.4+2i$, $p = -1.65+2i$, $\delta = 0.1$, $\alpha = 0.02$, $f = 0.5$, $h_2 = 0.3$ are shown in fig.4. Again, for sufficiently large group velocity a jump-transition from fast to slow pulses is observed. In the vicinity of the jump the trapped pulse becomes unstable to oscillations. It becomes more extended - in particular at its trailing end, where the concentration mode leads to an increase in the local growth rate - and phase slips arise. This is shown in fig.5. Here a pulse is depicted at the time

of the phase-slip process which is marked by the localized amplitude depression and the diverging local wave number. The phase slips lead to an oscillatory behavior of the velocity, the extrema of which are indicated by the dashed lines in fig.4. When the group velocity is increased further the trailing end develops larger peaks which then grow by themselves and eventually convection spreads throughout the system.

Experimentally, the only parameter that can be varied easily is the Rayleigh number, which enters the growth rate a_r. An interesting question is therefore whether the extended Ginzburg-Landau equations allow a hysteretic transition to fast pulses by changing a_r. Eq.(64) suggests this possibility, but is not valid for small velocities. This question is investigated for three cases.

The growth rate enters (64) in three places. The explicit dependence leads to an increase in velocity for all $f_r h_2 > 0$. So does the contribution (via the amplitude λ_0^2) from the damping term proportional to α. The diffusive term proportional to δ can, however, lead to a further decrease in the velocity with increasing amplitude. If inequalities (68,69) are satisfied, the pulse velocity is predicted to increase in the immediate vicinity of the saddle-node bifurcation and reach a value larger than s for $a_r = 0$. The result of a numerical simulation of this case is presented in fig.6 for $s = 0.3$, $d = 0.05 + 0.5i$, $f = 0.25$, c$=2.4 + i$, $p = -1.65$, $\alpha = 0.03$, $\delta = 0.1$, $h_2 = 0.3$. It shows indeed a hysteretic transition from a trapped pulse to an accelerated pulse. Interestingly, the trapped pulse becomes unstable to oscillations in the vicinity of the transition. Here it is the variation in the peak height of the concentration mode which leads to the oscillations in the velocity. Their minimal and maximal values are denoted by dotted lines.

Fig.7 shows the result of simulations for a case in which $c_r < 11d_r$ but inequality (69) is still satisfied (solid squares, $d_r = 0.3$, other parameters as in fig.6). As expected from the perturbation result the pulse velocity increases in the vicinity of the saddle-node bifurcation ($a_r \approx -0.5$), but it remains below the group velocity up to $a_r = -0.15$. For larger values of a_r the pulse becomes unstable. Finally, if both inequalities (68,69) are violated the perturbation analysis suggests that the pulse is slowed down even further with increasing a_r.

Two such cases are shown in fig.7 ($d_r = 0.6$). For $f_r = 0.35$ (open circles) and for $f_r = 0.25$ (open diamonds) the pulse is 'trapped' and its velocity *decreases* up to $a_r = -0.43$. For larger values of a_r the pulse starts to grow and eventually fills the whole cell. For weaker coupling to the concentration mode ($f_r = 0.2$, open triangles) the pulse velocity decreases with increasing a_r in the immediate vicinity of the saddle-node bifurcation, but for larger a_r the velocity starts to increase.

VI. CONCLUSION

In this paper I have investigated the influence of a long-wavelength mode on the dynamics of a traveling-wave pulse in the perturbed nonlinear Schrödinger equation. The study was motivated by experimental and numerical results on convection in binary liquids. It showed analytically that such a mode strongly affects the propagation velocity of such a pulse. In particular, two distinct mechanisms were identified which can lead to a 'trapping' of the pulse. As expected from numerical simulations of the full Navier-Stokes equations [21] this can be due to a reduction of the local growth rate of the traveling-wave mode. This effect arises, however, only at second order. Already at first order the frequency of the waves is affected by the long-wave mode which induces a change in the wave number and connected with it in the velocity of the pulse. This effect was neither anticipated by the numerical simulations nor by the experiments. For simplified boundary conditions both effects are found to reduce the velocity unless dispersion is very large. Within the framework presented here hysteretic transitions can occur between 'trapped' pulses and fast pulses when the local growth rate (i.e. the Rayleigh number) is increased. So far no such transitions have been found in the experiments or the numerical simulations of the Navier-Stokes equations.

An increase in the Rayleigh number in the experiments often leads to long pulses which cannot be regarded as perturbed solitons anymore. Instead they can be considered as a bound state of two fronts. The dynamics of such fronts is also strongly affected by the long-wave mode. In particular, within the extended Ginzburg-Landau equations discussed here

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it has been found that their interaction can change from attractive to repulsive when the direction of propagation of the pulse changes sign [39–41]. Thus, backward traveling pulses are stable and pulses of different sizes can coexist in agreement with experimental results [35]. Further studies show [41] that the coexistence of stable pulses found in the numerical simulations [21] can also be understood within this framework.

No attempt has been made so far to investigate the effect of increasing the Lewis number (i.e. the mass diffusion) using full numerical simulations of the Navier-Stokes equations. According to the present analysis the pulse velocity should increase drastically when the Lewis number is increased. Such an investigation would provide a stringent test for the relevance of the extended Ginzburg-Landau equations to binary-mixture convection.

There have been attempts to find stable localized pulses in two-dimensional convection. To date these pulses have not proved to be stable but decay after long transients [19]. The origin of this instability is not understood. In addition to the long-wave concentration mode a long-wave vorticity mode (mean flow) is expected to be relevant. A related interesting problem is the dynamics of two-dimensional propagating fronts. Since the concentration mode slows down certain types of fronts [40] one may expect that it could lead to a transverse instability of such fronts similar to the Mullins-Sekerka instability in solidification [20].

The relevance of the equations discussed in this paper is not confined to convection in binary liquids. Equations of the same form have also been derived for waves on the interface between two immiscible liquids flowing in a channel [43]. It is conceivable that nonlinear gradient terms arising in the description of other wave systems are also due to the elimination of a second mode (e.g. [58,59]). In regimes in which such a mode becomes slow as compared to the wave it would have to be described by a separate dynamical equation and could have similarly strong effects on the waves as found in the present paper.

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APPENDIX A: DERIVATION OF THE EXTENDED GINZBURG-LANDAU EQUATIONS

In this appendix I rederive the extended Ginzburg-Landau equations (ECGL) under slightly more general conditions than previously [38]. In the previous derivation the Lewis number L and the growth rate a of the convective mode were taken to be of the same order. In the limit of very small growth rate, i.e. for $|a| \ll L$, the concentration mode C can be eliminated adiabatically from the resulting equations to yield the usual complex Ginzburg-Landa equation (CGL). However, with the coefficients calculated in [38] one does not recover the correct values of the coefficients in the CGL, not even their values in the limit of small Lewis number. The generalization presented here overcomes this problem and agrees for $|a| \ll L$ quantitatively with the CGL. At the same time it shows that the terms $|A|^2$ and $i\left(A^* \partial_x A - A \partial_x A^*\right)$, which in the previous derivation were found to have vanishing coefficients at quadratic and third order, respectively, become relevant at higher order.

A simple analysis of (1,2) shows that the coefficients derived in [38] do not lead to the correct coefficients of the single CGL if the concentration mode is eliminated adiabatically for $|a| \ll L \propto \alpha$. This is due to the fact that any L-dependence of the nonlinear coefficients is shifted to higher-order terms due to the initial assumption $L = O(\epsilon^2) = O(a)$. Since in the adiabatic elimination the contributions from the nonlinear terms are divided by L, higher-order terms of the form $L|A|^2$ become relevant. The goal is to obtain certain of these contributions without performing the full expansion to the corresponding high order.

As pointed out in sec.II, (1,2) are reconstituted equations [42]; i.e. the solvability conditions arising at $O(\epsilon)$, $O(\epsilon^2)$ and $O(\epsilon^3)$ are not satisfied separately but are combined into a single solvability condition. One could imagine such a reconstitution involving all orders in ϵ. The higher-order terms would then have the form $L^n A^m C^p$ and suitable spatial derivatives thereof. They could be summed up into a term $A^m C^p$ (and the respective spatial derivatives) with an L-dependent coefficient. Alternatively, one could obtain this L-dependence directly by avoiding the expansion in L, thus providing a summation of certain contributions in L

to all orders. This is the approach taken here. A more detailed analysis is then required to identify for which of the coefficients the procedure restores the complete L-dependence.

Within the Boussinesq-approximation the basic equations for two-dimensional convection in a binary mixture at infinite Prandtl number read [32,33]

$$
\begin{pmatrix}
-\Delta^2 & -(1+\psi)\partial_x & -\psi\partial_x \\
-\mathcal{R}\partial_x & \partial_t - \Delta & 0 \\
-\mathcal{R}\partial_x & \partial_t & \partial_t - L\Delta
\end{pmatrix}
\begin{pmatrix}
\phi \\ \theta \\ \eta
\end{pmatrix}
=
\begin{pmatrix}
0 \\
(\partial_z\phi\partial_x - \partial_x\phi\partial_z)\theta \\
(\partial_z\phi\partial_x - \partial_x\phi\partial_z)(\eta + \theta)
\end{pmatrix}
\tag{A1}
$$

where ϕ denotes the stream function, θ the deviation of the temperature from the conductive state and $\eta = (c/\psi - \theta)$ with c being the deviation of the concentration field from the conductive state. The separation ratio is denoted by ψ and $\mathcal{R}$ and L are the Rayleigh number and the Lewis number, respectively. For free-slip-permeable boundary conditions the fields are expanded as [38]

$$
\begin{aligned}
\phi = {}& \epsilon\, e^{i\,q\,x - i\omega_h t} \sin(\pi z)\, A + \epsilon^2 \{ \sin(2\pi z)\, (\tfrac{1}{2}D_\phi + E_\phi\, e^{i\,q\,x - i\omega_h t}) + \\
& \sin(\pi z)\left(\tfrac{1}{2}F_\phi + (G_1 + G_2 + G_3)\, e^{i\,q\,x - i\omega_h t}\right) + \sin(3\pi z)\, H_\phi\, e^{i\,q\,x - i\omega_h t}\} \\
& + c.c. + O(\epsilon^3),
\end{aligned}
\tag{A2}
$$

$$
\begin{aligned}
\theta = {}& \epsilon\, e^{i\,q\,x - i\omega_h t} \sin(\pi z)\, A\, \zeta_1 + \epsilon^2 \{ \sin(2\pi z)\left(\tfrac{1}{2}D_\theta + E_\theta\, e^{i\,q\,x - i\omega_h t}\right) \\
& + \sin(\pi z)\left(\tfrac{1}{2}F_\theta + \zeta_1\,(G_1 - G_2 + G_3)\, e^{i\,q\,x - i\omega_h t}\right) + \sin(3\pi z)\, H_\theta\, e^{i\,q\,x - i\omega_h t}\} \\
& + c.c. + O(\epsilon^3),
\end{aligned}
\tag{A3}
$$

$$
\begin{aligned}
\eta = {}& \epsilon\left(e^{i\,q\,x - i\omega_h t} \sin(\pi z)\, A\, \zeta_2 + C \sin(2\pi z)\right) + \epsilon^2 \{ \sin(2\pi z)\left(\tfrac{1}{2}D_\eta + E_\eta\, e^{i\,q\,x - i\omega_h t}\right) + \\
& \sin(\pi z)\left(\tfrac{1}{2}F_\eta + \zeta_2\,(G_1 + G_2 - G_3)\, e^{i\,q\,x - i\omega_h t}\right) + \sin(3\pi z)\, H_\eta\, e^{i\,q\,x - i\omega_h t}\} \,) \\
& + c.c. + O(\epsilon^3).
\end{aligned}
\tag{A4}
$$

The Rayleigh number is expanded as

$$
\mathcal{R} = R_0(q) + \epsilon R_1 + \epsilon^2 R_2 \qquad \text{with} \qquad R_0 = \frac{(q^2 + \pi^2)^3\,(1+L)}{q^2\,(1+\psi)}.
\tag{A5}
$$

At this point the expansion wave number q is performed is left undetermined. This allows a consistency check of the resulting coefficients since certain higher-order coefficients have to

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correspond to derivatives of lower-order coefficients with respect to the wave number q (see appendix B). Eventually q will be chosen to be the critical wave number $q_c = \pi/\sqrt{2}$. The separation ratio ψ and the Hopf frequency are expressed as

$$\psi = -\frac{\Omega^2}{\Omega^2 + 1}, \qquad \omega_h = \omega(q^2 + \pi^2) \text{ with } \omega = \sqrt{\Omega^2 - L^2 + L\Omega^2}. \tag{A6}$$

The linear eigenvector $(1, \zeta_1, \zeta_2)^t$ of the convective mode is given by

$$\zeta_1 = \frac{i(\Omega^2 + 1)(1 + L)(q^2 + \pi^2)^2}{(1 - i\omega)q}, \qquad \zeta_2 = \zeta_1 \frac{1}{L - i\omega}. \tag{A7}$$

As mentioned above, the Lewis number L is not expanded. Thus, already at $O(\epsilon)$ a damping term $-4\pi^2 LC$ arises formally in the solvability condition for C. At this order no time-derivative arises which would balance this term. Since L is eventually taken to be small $(O(\epsilon))$ it is reasonable to defer the balancing until all solvability conditions for C are reconstituted into a single equation.

At $O(\epsilon^2)$ one obtains

$$F_\theta = 0, F_\phi = 0, F_\eta = 0, \qquad E_\theta = 0, E_\phi = 0, E_\eta = 0, \tag{A8}$$

$$D_\theta = \frac{1}{2} \frac{(q^2 + \pi^2)^2 (\Omega^2 + 1) |A|^2}{(L - \Omega^2 - 1)\pi}, \qquad D_\phi = \frac{1}{16} \frac{\Omega^2 \left(\frac{\partial}{\partial X} C\right)}{\pi^4 (\Omega^2 + 1)}, \tag{A9}$$

$$G_2 = \frac{1}{4} \frac{q^2 (iL - \omega) A R_1}{(1 + L)\omega (q^2 + \pi^2)^3 (\Omega^2 + 1)} - \frac{1}{4} \frac{i\Omega^2 \pi q^2 C A}{\omega (\Omega^2 + 1)(q^2 + \pi^2)^3}$$
$$- \frac{1}{2} \frac{(-\pi^2 L + 2 L q^2 - i q^2 \omega)\left(\frac{\partial}{\partial X} A\right)}{\omega q (q^2 + \pi^2)}, \tag{A10}$$

$$G_3 = \frac{1}{4} \frac{i q^2 A R_1}{\omega (\Omega^2 + 1)(q^2 + \pi^2)^3} - \frac{1}{4} \frac{\pi q^2 (\omega + iL) C A}{\omega (\Omega^2 + 1)(q^2 + \pi^2)^3}$$
$$- \frac{1}{2} \frac{(-\pi^2 - \pi^2 L + 2 q^2 + 2 L q^2 - 3 i q^2 \omega + i\omega \pi^2)\left(\frac{\partial}{\partial X} A\right)}{\omega q (q^2 + \pi^2)}, \tag{A11}$$

$$H_\phi = 4 \frac{i(19i + 3\omega)\Omega^2 A C}{(\Omega^2 + 1)\pi^3 k_1}, \tag{A12}$$

$$H_\eta = -\frac{i\sqrt{2} A C (27 L - 6832 + 1083 i\omega)}{k_1}, \tag{A13}$$

$$H_\theta = -27 \frac{i(1 + L)\sqrt{2}\Omega^2 A C}{k_1} \tag{A14}$$

with

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$$k_1 = 513 \left(L^2 - \Omega^2 - L\Omega^2\right) + 3249\,\omega^2 + 20496\,i\omega(L+1) - 129808\,L. \tag{A15}$$

Note that for simplicity the expressions H_ϕ, H_θ and H_η have been given only for $q = q_c$. In the expressions for G_2 and G_3, however, the q-dependence has been retained for use in appendix B. The amplitudes G_1 and D_η can be set to 0 since they only renormalize the amplitudes A and C, respectively. Combining the solvability conditions for A and C arising at $O(\epsilon)$, $O(\epsilon^2)$ and $O(\epsilon^3)$ one obtains the ECGL

$$\partial_t A + (s + s_2 C)\partial_x A = d\partial_x^2 A + (a + fC + f_2 C^2 + f_3\partial_x C)A + cA|A|^2, \tag{A16}$$

$$\partial_t C = \delta\partial_x^2 C - \alpha C + h_1|A|^2 + h_2\partial_x|A|^2 + h_3 C|A|^2 + \tag{}$$

$$ih_4(A^*\partial_x A - A\partial_x A^*). \tag{A17}$$

The coefficients are given by

$$a = \frac{1}{9}\frac{\epsilon\,(\omega + iL - i\Omega^2)\,R_2}{\pi^2\,\omega\,(\Omega^2 + 1)}, \tag{A18}$$

$$s = \sqrt{2}\,\omega\,\pi, \qquad s_2 = -\frac{2}{27}\frac{(i\omega - 1)\,\Omega^2\,\sqrt{2}\,\epsilon}{(\Omega^2 + 1)\,\pi^2\,\omega}, \tag{A19}$$

$$d = 2\,(1 + L)\,\epsilon + \frac{i\,(L^2 - L\Omega^2 + 2L - \Omega^2)\,\epsilon}{\omega}, \tag{A20}$$

$$c = -\frac{1}{24}\frac{(4L^3 - L^2\Omega^2 - 3\Omega^4 L - 3\Omega^4)\,\epsilon\,\pi^2}{\Omega^2\,\omega^2\,(L - \Omega^2 - 1)} + \frac{1}{24}\frac{i\,(4L^2 + 3L\Omega^2 - 3\Omega^4)\,\pi^2\,\epsilon}{\Omega^2\,\omega\,(L - \Omega^2 - 1)}, \tag{A21}$$

$$f = -\frac{1}{27}\frac{3\Omega^4 + L^2\Omega^2 + 3\Omega^4 L - 4L^3}{\pi\,\omega^2\,(\Omega^2 + 1)} - \frac{1}{27}\frac{i\,(4L^2 + 3\Omega^2)}{\pi\,(\Omega^2 + 1)\,\omega}, \tag{A22}$$

$$f_2 = \frac{1}{972}\frac{i\,\Omega\,(255\,\Omega^2 + 1651\,i\,\Omega + 1708)\,(\Omega + i)\,\epsilon}{\pi^4\,(\Omega^2 + 1)^2\,(+427\,i + 57\,\Omega)}$$

$$-\frac{1}{1944}\frac{i\,\epsilon\,\Omega\,(14535\,\Omega^4 + 240342\,i\,\Omega^3 - 1119241\,\Omega^2 - 778848\,i\,\Omega - 1263066)}{\pi^4\,(\Omega^2 + 1)^2\,(427\,i + 57\,\Omega)^2}\,L$$

$$+O(L^2), \tag{A23}$$

$$f_3 = -\frac{1}{864}\frac{(32L^2 - 32\,i\,\omega\,L - 32\,L\Omega^2 + 48\,\Omega^2 - 21\,i\,\omega\,\Omega^2)\,\sqrt{2}\,\epsilon}{(\Omega^2 + 1)\,\omega\,\pi^2}, \tag{A24}$$

$$\alpha = 4\,\pi^2\,\frac{L}{\epsilon}, \qquad \delta = \epsilon\,\frac{27}{64}\left(\Omega^2\,(1 + L) + \frac{64}{27}L\right), \tag{A25}$$

$$h_1 = -\frac{9}{2}\frac{\pi^5\,(\Omega^2 + 1)\,L}{\Omega^2\,(\Omega^2 + 1 - L)}, \qquad h_2 = -\frac{3}{8}\frac{\epsilon\,(\Omega^2 + 1)\,\pi^4\,(5\Omega^2 + 2)\,\omega\,\sqrt{2}}{\Omega^2\,(\Omega^2 + 1 - L)}, \tag{A26}$$

$$h_3 = \frac{1}{12}\frac{(\Omega^2 + 4)\,\pi^2\,\epsilon}{\Omega^2 + 1} -$$

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$$\frac{1}{36} \epsilon \, \pi^2 \frac{574389 \, \Omega^6 + 28147427 \, \Omega^4 + 51487378 \, \Omega^2 + 25526060}{\Omega^2 \, (\Omega^2 + 1)^2 \, (3249 \, \Omega^2 + 182329)} L + O(\, L^2\,), \qquad \text{(A27)}$$

$$h_4 = -\frac{3}{2} \frac{\epsilon \, (\Omega^2 + 1) \, \pi^4 \, L \, \sqrt{2}}{\Omega^2 \, (L - \Omega^2 - 1)}. \qquad \text{(A28)}$$

In these expressions q has been set equal to q_c. Note that h_1 and h_4 are then proportional to the Lewis number and are therefore of higher order in agreement with the previous calculation [38].

Without a further discussion it is not clear to which order in L the coefficients derived above are actually correct; in general the reconstitution does not yield *all* contributions. Consider the expansion with L taken explicitly to be $O(\epsilon)$, i.e. $L = \epsilon L_1$. The procedure involves essentially three steps. First the equations are expanded in a straightforward way. For the relevant modes A and C this yields evolution equations containing also the other, stable Fourier modes like G_i etc. In a second step the stable modes are eliminated adiabatically using equations like (A8)-(A14)[4]. This leads to equations of the form

$$\partial_t A = A \, \mathcal{F}(|A|^2, C, \partial_x, \partial_t, L_1), \qquad \partial_t C = \mathcal{G}(|A|^2, C, \partial_x, \partial_t, L_1), \qquad \text{(A29)}$$

where the form of the equation is dictated by symmetry. The functions $\mathcal{F}$ and $\mathcal{G}$ stand for invariant polynomials in their arguments. The notation indicates that A and A^* appear only with equal powers as required by translation symmetry. The dependence of $\mathcal{F}$ and $\mathcal{G}$ on R_1 is not indicated. Many of the high-order terms differ from the lower-order terms only in their powers in L_1. By not expanding in L all these terms are summed up into an L-dependent coefficient of the lowest-order term with the same structure regarding the amplitudes A and C and their derivatives.

In a third step the time-derivatives in the nonlinear terms are replaced recursively using (A29). If only amplitudes and derivatives are counted, this introduces terms of lower order since the evolution equations have the form[5]

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[4]I ignore in this discussion the possible effect of other critical concentration modes.

[5]Of course, the coefficients a and $\alpha \propto L$ are small; therefore the true order of the respective terms

$$\partial_t A = aA + ..., \qquad \partial_t C = -\alpha C + ... \tag{A30}$$

As a consequence, the replacement of $\partial_t A$ contributes to the R_1-dependence $(a \propto R_1)$ and the replacement of $\partial_t C$ to the L-dependence of the corresponding lower-order term. Even if no expansion in L is performed, the latter contributions are not obtained unless the initial expansion is taken all the way to the order of the term involving the time-derivative. Thus, due to the third step the above procedure will not sum up the L-dependence to all orders unless the coefficient in question has no contribution from higher-order terms containing time-derivatives $\partial_t C$.

Since the elimination of $\partial_t C$ in a given term introduces a factor C, which cannot be eliminated by further substitutions, low-order coefficients of terms containing only factors of A and its derivatives will not be affected by the replacement of $\partial_t C$ in the higher-order terms and are expected to be correct to all orders in L. Coefficients of terms containing a factor C, however, may be affected and may therefore only be correct to the order of the initial expansion. The terms omitted in (A30) do not affect this argument since they do not decrease the order in terms of A, C and their derivatives. For instance, they lead to replacements like $\partial_t A \to -s\partial_x A$ or $\partial_t C \to h_1|A|^2$.

The above arguments suggest that the coefficients s, c, h_1, h_2, and h_4 as given in (A18)-(A28) are correct to all orders in L. It is therefore expected that the adiabatic elimination of C for $|a| \ll L$ will yield the same value for the cubic coefficient of A as found in the direct derivation of the CGL.

APPENDIX B: ELIMINATION OF C AND WARD IDENTITIES

In this appendix it is demonstrated that the coefficients as calculated in appendix A are consistent with those of the usual CGL in the appropriate limit and satisfy the relevant Ward identities [60].

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remains the same.

To show the consistency with the usual CGL, C is adiabatically eliminated for $\epsilon \ll L$. In the previous expansion [38] this did not lead to the correct cubic coefficient in the CGL since for $\epsilon \ll L$ higher-order terms of the form $L|A|^2$ become relevant as well, which are of higher order in the previous expansion. They are kept in the expansion presented in appendix A. To wit, h_1 is of $O(L)$ and vanished therefore in the previous calculation. In the present approach it gives an $O(1)$ contribution to C,

$$C_{ad} = \frac{9}{8} \frac{|A|^2 \pi^3 (\Omega^2 + 1)}{\Omega^2 (L - \Omega^2 - 1)} + O(\partial_x |A|^2). \tag{B1}$$

Inserting C_{ad} into (A16) gives an additional cubic term in A. The two cubic terms combined yield the purely imaginary cubic coefficient of the usual Ginzburg-Landau equation

$$c_{ad} = -\frac{i}{8} \frac{\pi^2}{\omega}, \tag{B2}$$

in agreement with previous work [61].

An additional test of the coefficients calculated in appendix A is possible using the Ward identities which relate coefficients of gradient terms with q-derivatives of other, lower-order coefficients [60]. These identities arise due to the fact that the Ginzburg-Landau equations can be derived by an expansion around any wave number q on the neutral curve. Of course, in an expansion around a wave number different than that corresponding to the minimum of the neutral curve the group velocity parameter s is complex and the expansion of the Rayleigh number contains also the linear term R_1 (cf. (A5)). Spatially periodic solutions with wave numbers $q_1 \equiv q + \Delta q$ differing slightly from q can then be determined in two ways. Either q is changed to q_1 or the difference Δq in wave number is absorbed in the slow wave number Q. In the former case the coefficients in the Ginzburg-Landau equation change. In the latter case the gradient terms come into play. Any physical quantity P like the stream function ϕ depends on the expansion wave number q as well as the slow wave number Q. Since both approaches have to lead to the same final result one obtains the invariance condition

$$\frac{d}{dQ} P(q - \epsilon Q, Q) = 0. \tag{B3}$$

For definiteness, consider a specific Fourier mode of the stream function ϕ (cf. (A2)),

$$\phi_{q,\omega_h} = e^{i(q-\epsilon Q)x - i(\omega_h + \epsilon \hat{\Omega})t} \sin(\pi z) \left(\epsilon \mathcal{A} e^{iQX + i\hat{\Omega}T} + \epsilon^2 (G_1 + G_2 + G_3) + O(\epsilon^3) \right). \tag{B4}$$

From (A10,A11) one finds

$$G_i = e^{iQX + i\hat{\Omega}T} g_i, \qquad \text{with} \qquad g_i = const. \tag{B5}$$

Since G_1 can be set to zero one obtains

$$\frac{d}{dQ}\mathcal{A} = -\epsilon \frac{d}{dQ}(g_2 + g_3) = \epsilon \rho \mathcal{A} + O(\epsilon^2) \tag{B6}$$

with

$$\rho = -\frac{(\pi^2 - 3q^2)}{(q^2 + \pi^2)q}. \tag{B7}$$

The concentration mode C is independent of Q since it is not the amplitude of a periodic function.

Conditions for the coefficients in the extended Ginzburg-Landau equations (A16,A17) arise from taking the Q-derivative of (A16,A17) and inserting the invariance condition (B6). It is important to note that the derivative has to be taken at fixed physical parameters like the Rayleigh number. Thus, R_1 has to be replaced by $(\mathcal{R} - R_0(q))/\epsilon$. This implies that the linear coefficient $a(q)$, which is proportional to R_1, satisfies

$$\frac{d}{dQ}a(q - \epsilon Q) \equiv -\epsilon \frac{d}{dq}a(q - \epsilon Q) = O(1), \tag{B8}$$

whereas the Q-derivative of all other coefficients is $O(\epsilon)$. Applied to the equation for the convective amplitude, (A16), the invariance condition leads then to the expected relations

$$s = i\epsilon \frac{d}{dq}a,, \qquad s_2 = i\frac{d}{dq}f. \tag{B9}$$

The Q-derivative of (A17) yields

$$2h_4 = \frac{d}{dq}h_1 - (\rho + \rho^*)h_1 + O(\epsilon). \tag{B10}$$

At first sight it might be surprising that h_4 is not just proportional to the derivative of h_1. However, since the eigenvector $(1, \zeta_1, \zeta_2)^t$, which connects the amplitude A with the physical quantities, depends explicitly on the wave number q additional terms like $(\rho + \rho^*)h_1$ have to be expected in general.

To check the Ward identities (B9,B10) the q-dependence of a, f and h_1 has to be kept. The calculation presented in appendix A yields

$$a(q) = -\frac{1}{8} \frac{i\, R_1{}^2\, \Omega^2\, (L - \Omega^2 - 1)\, q^4\, \epsilon}{\omega^3\, (\Omega^2 + 1)^2\, (q^2 + \pi^2)^5} + \frac{1}{2} \frac{q^2\, (\omega + i\, L - i\, \Omega^2)}{\omega\, (\Omega^2 + 1)\, (q^2 + \pi^2)^2}\, (R_1 + \epsilon R_2),$$
(B11)

$$f(q) = \frac{1}{2} \frac{(2\, \pi^2\, L^3 + q^2\, L^2\, \Omega^2 - \Omega^4\, \pi^2\, L - q^2\, \Omega^4\, L - \Omega^2\, L^2\, \pi^2 - q^2\, \Omega^4 - \pi^2\, \Omega^4)\, \pi\, q^2}{\omega^2\, (\Omega^2 + 1)\, (q^2 + \pi^2)^3} +$$
$$i\, \left(\frac{1}{4} \frac{R_1\, \epsilon\, \Omega^2\, (\Omega^2 + 1 - L)\, L\, \pi\, q^4}{\omega^3\, (q^2 + \pi^2)^5\, (\Omega^2 + 1)^2} + \frac{1}{2} \frac{(2\, L^2\, \pi^2 + \Omega^2\, \pi^2 + q^2\, \Omega^2)\, \pi\, q^2}{(\Omega^2 + 1)\, \omega\, (q^2 + \pi^2)^3} \right),$$
(B12)

$$h_1(q) = -\frac{1}{2} \frac{\epsilon\, R_1\, q^2\, (-\pi^2\, L\, \Omega^2 + L\, q^2\, \Omega^2 - 2\, \pi^2 - \Omega^2\, \pi^2 + q^2\, \Omega^2)}{\Omega^2\, (1 + L)\, (L - \Omega^2 - 1)\, (q^2 + \pi^2)\, \pi}$$
$$+ 2 \frac{\pi\, (\Omega^2 + 1)\, (q^2 + \pi^2)^2\, L}{\Omega^2\, (L - \Omega^2 - 1)},$$
(B13)

where R_1 and R_2 have to be replaced by $\mathcal{R}$ and $R_0(q)$ via $\mathcal{R} = R_0(q) + \epsilon R_1 + \epsilon^2 R_2$. The values for s, s_2 and h_4 obtained by inserting (B11,B12,B13) into (B9,B10) agree with those obtained in (A19) and (A28) up to terms of order $O(L^2)$ and $O(\epsilon)$ for s_2 and h_4, respectively. These differences are expected based on the discussion of the correctness of the L-dependence of s_2 in appendix A and in view of (B10).
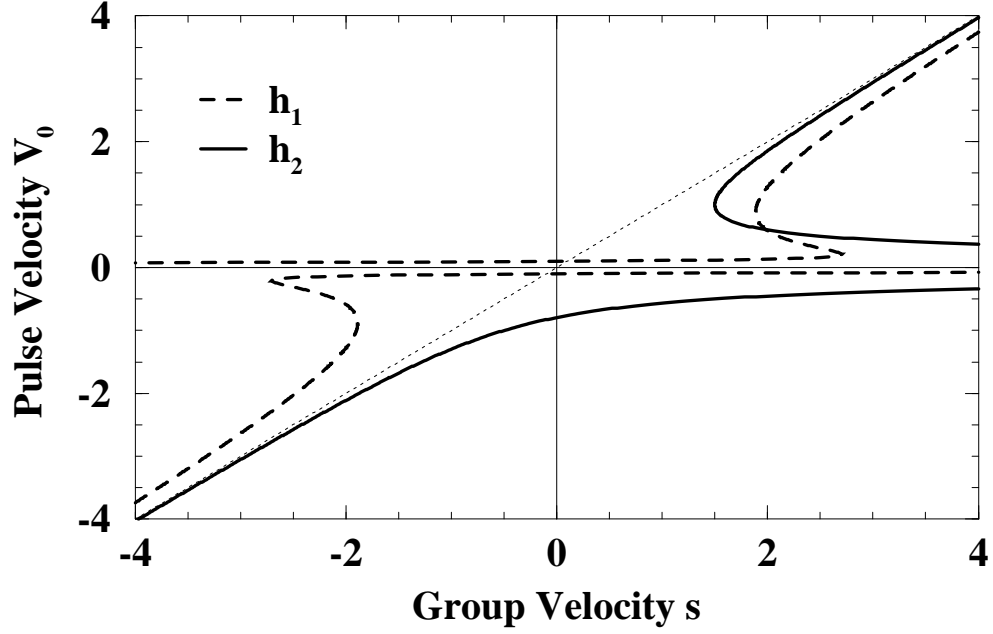
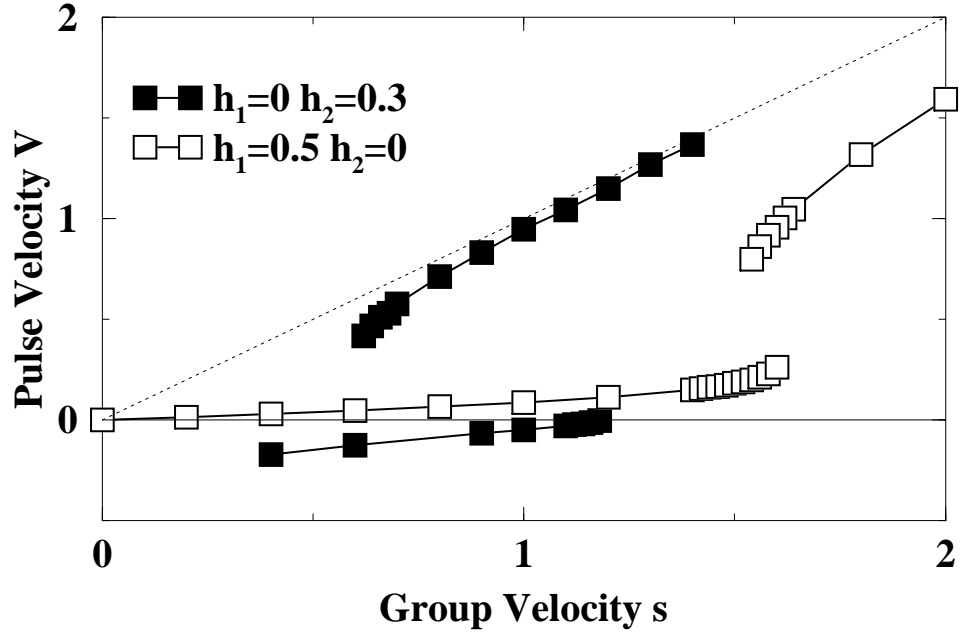FIG. 1. Sketch of the contributions to the pulse velocity V_0 from the h_1- and the h_2-term (cf. (55)).

FIG. 2. Numerically determined pulse velocity V as a function of the group velocity s for $d_r = 0.15 + i$, $a = -0.24$, $f = 0.5i$, $c = 2.4 + 2i$, $p = -1.65 + 2i$, $\delta = 0.1$, $\alpha = 0.08$ and the indicated values of h_1 and h_2.

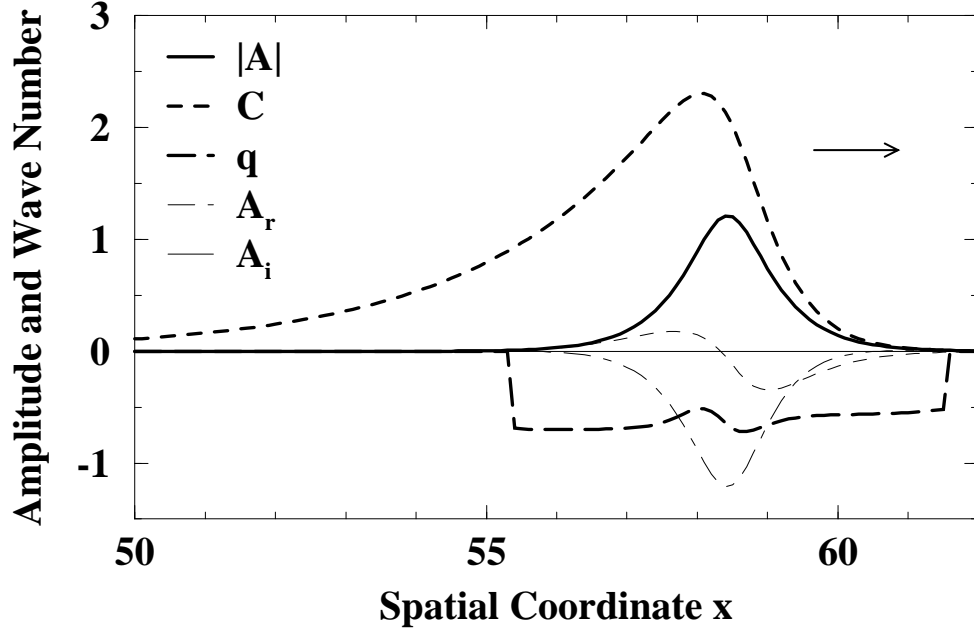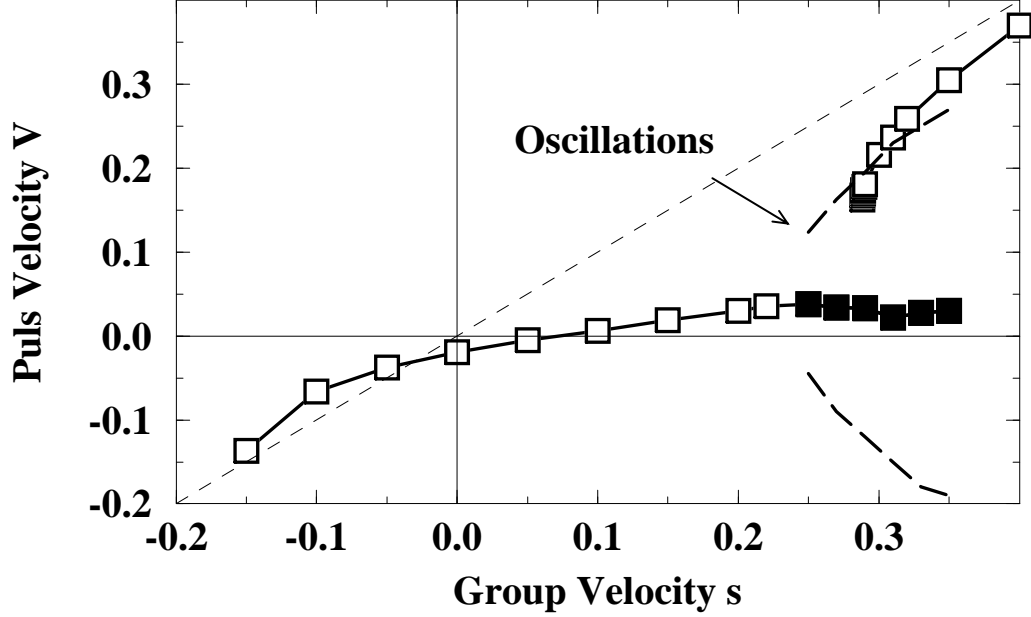FIG. 3. Typical pulse solution for $f_i \neq 0$ and $h_1 \neq 0$. The dependence of the frequency on the spatially varying asymmetric concentration leads to a shift in the wave number (cf. (52)) which changes the velocity of the pulse (cf. (23)).

FIG. 4. Numerically determined pulse velocity V as a function of the group velocity s for $a = -0.24$, $d = 0.15 + i$, $c = 2.4 + 2i$, $p = -1.65 + 2i$, $\delta = 0.1$, $\alpha = 0.02$, $f = 0.5$, $h_2 = 0.3$. The dashed lines indicate the minimal and maximal values of V in the oscillatory regime (solid squares).
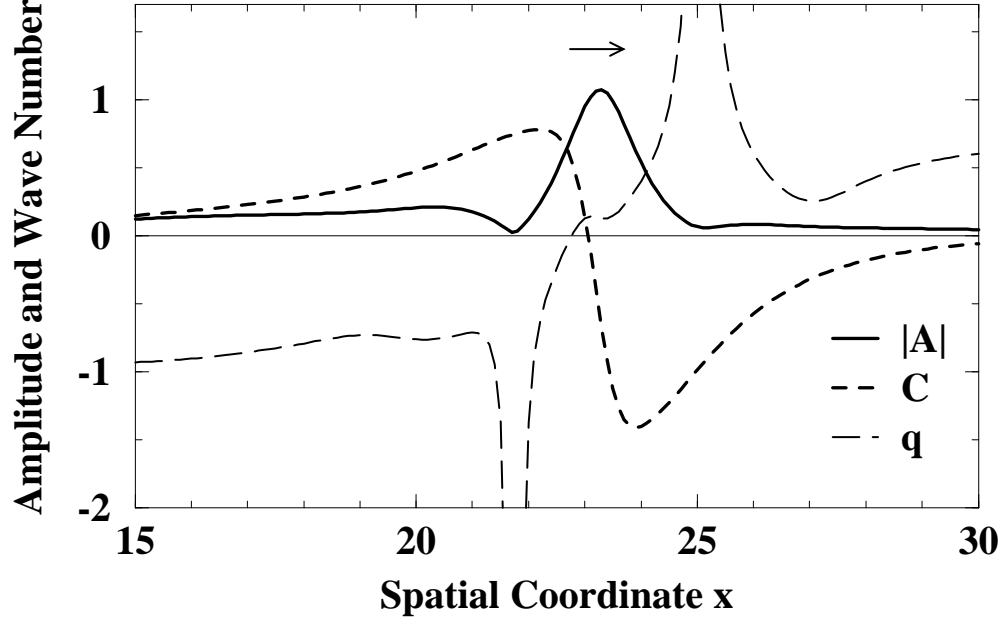
FIG. 5. Pulse solution for $s = 0.35$ (other parameters as in fig.4). The depression in $|A|$ and the diverging wave number indicate that a phase slip has just occurred. These phase slips lead to an oscillatory behavior of the velocity.
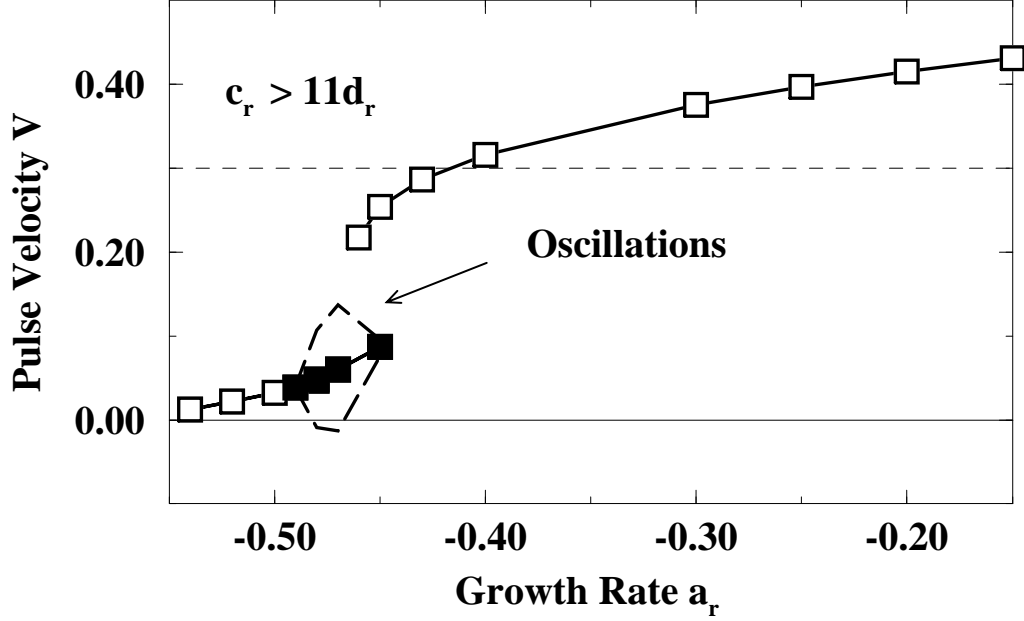

FIG. 6. Numerically determined pulse velocity V as a function of the growth rate a_r for $s = 0.3$, $d = 0.05 + 0.5i$, $f = 0.25$, $c = 2.4 + i$, $p = -1.65$, $\alpha = 0.03$, $\delta = 0.1$, $h_2 = 0.3$. The dashed lines indicate the minimal and maximal values of V in the oscillatory regime (solid squares).
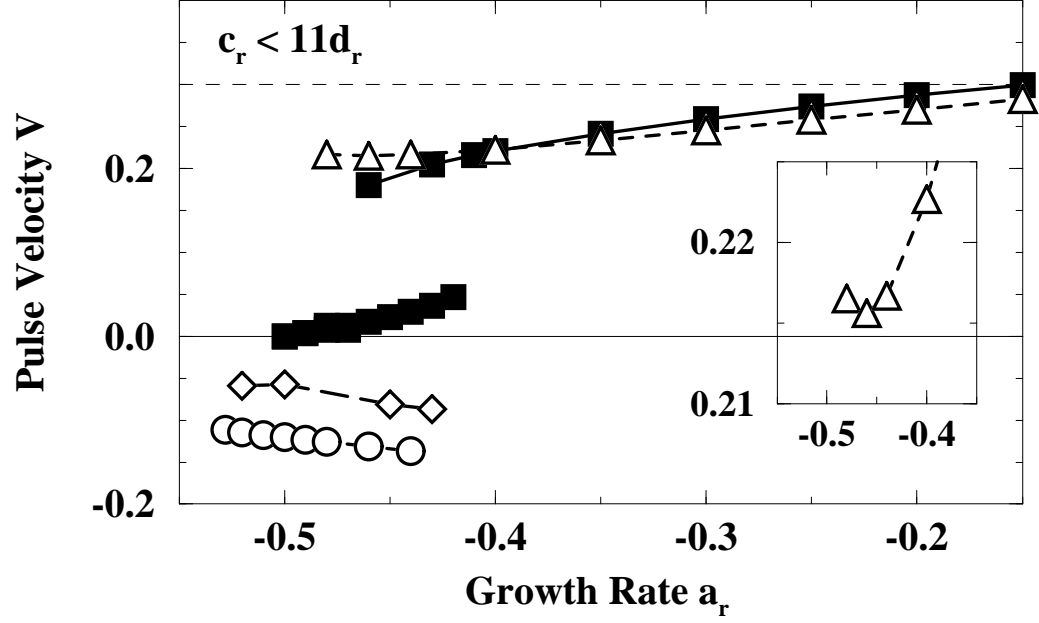
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FIG. 7. Numerically determined pulse velocity V as a function of the growth rate a_r for various values of d_r and f_r: $d_r = 0.3$, $f_r = 0.25$ (solid squares), $d_r = 0.6$, $f_r = 0.2$ (open triangles), $d_r = 0.6$, $f_r = 0.25$ (open diamonds), $d_r = 0.6$, $f_r = 0.35$ (open circles). Other parameters as in fig.6. Inset shows initial decrease in velocity for $d_r = 0.6$, $f_r = 0.2$ as expected from (68).